27-th Canadian Mathematical Olympiad 1995

1. Let $f(x) = \dfrac{9^x}{9^x + 3}$. Evaluate the sum

$$f\left(\frac{1}{1996}\right) + f\left(\frac{2}{1996}\right) + \cdots + f\left(\frac{1995}{1996}\right).$$

2. Let a, b, c be positive real numbers. Prove that

$$a^a b^b c^c \geq (abc)^{\frac{a+b+c}{3}}.$$

3. We call a non-convex quadrilateral without self-intersections a *boomerang*. Suppose that the interior of a convex s-gon C is decomposed into the union of q quadrilaterals with disjoint interiors, b of which are boomerangs. Show that $q \geq b + \dfrac{s-2}{2}$.

4. Let $n > 0$ and $k \geq 0$ be given integers. Show that the equation

$$x_1^3 + x_2^3 + \cdots + x_n^3 = y^{3k+2}$$

has infinitely many solutions in positive integers x_i and y.

5. Suppose that u is a real parameter with $0 < u < 1$. Define

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq u, \\ 1 - \left(\sqrt{ux} + \sqrt{(1-u)(1-x)}\right)^2 & \text{if } u \leq x \leq 1, \end{cases}$$

and define the sequence u_n recursively by $u_1 = f(1)$ and $u_n = f(u_{n-1})$ for all $n > 1$. Show that there exists a positive integer k for which $u_k = 0$.

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