14-th Canadian Mathematical Olympiad 1982

May 5, 1982

1. Let O be a point in the plane of a convex quadrilateral $A_1A_2A_3A_4$ and let B_1, B_2, B_3, B_4 be points such that OB_i is parallel and equal in length to A_iA_{i+1} for $i = 1,2,3,4$ (where $A_5 = A_1$). Show that the area of quadrilateral $B_1B_2B_3B_4$ is twice that of $A_1A_2A_3A_4$.

2. Let a,b,c be the roots of the equation $x^3 - x^2 - x - 1 = 0$.

   (a) Show that a,b,c are distinct.

   (b) Show that $\dfrac{a^{1982} - b^{1982}}{a-b} + \dfrac{b^{1982} - c^{1982}}{b-c} + \dfrac{c^{1982} - a^{1982}}{c-a}$ is an integer.

3. Determine the smallest number $g(n)$ os points of a set in the n-dimensional Euclidean space $\mathbb{R}^n$ such that every point in $\mathbb{R}^n$ is at irrational distance from at least one point in that set.

4. Let f_n be the number of permutations of the set $S_n = \{1,2,\dots,n\}$ having no fixed points, and g_n be the number with exactly one fixed point. Show that $|f_n - g_n| = 1$.

5. The altitudes of a tetrahedron $ABCD$ rom A,B,C and D have lengths h_a, h_b, h_c, h_d respectively. These altitudes are are extended externally to points A', B', C', D' respectively, where $AA' = k/h_a$, $BB' = k/h_b$, $CC' = k/h_c$ and $DD' = k/h_d$ for some constant k. Prove that the centroids of the tetrahedrons $ABCD$ and $A'B'C'D'$ coincide.

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