6-th Canadian Mathematical Olympiad 1974

Part A

1. (a) If $x=(1+1/n)^n$ and $y=(1+1/n)^{n+1}$, show that $x^y=y^x$.

   (b) Show that, for all $n\in\mathbb{N}$, $\sum_{k=1}^{n}(-1)^{k+1}k^2=(-1)^{n+1}\sum_{k=1}^{n}k$.

2. Let $ABCD$ be a rectangle with $BC=3AB$, and P,Q be the points on BC such that $BP=PQ=QC$. Prove that $\angle DBC+\angle DPC=\angle DQC$.

Part B

3. Consider the polynomial $f(x)=a_0+a_1x+\cdots+a_nx^n$, where $0\le a_i\le a_0$ for $i=1,2,\dots,n$. If $f(x)^2=b_0+b_1x+\cdots+b_{n+1}x^{n+1}+\cdots+b_{2n}x^{2n}$, prove that $b_{n+1}\le\frac{1}{2}f(1)^2$.

4. Let n be a positive integer. To any n real numbers satisfying x_i satisfying $0\le x_i\le 1$, $i=1,2,\dots,n$, we correspond the sum $\sum_{1\le i<j\le n}|x_i-x_j|$. Find the largest possible value $S(n)$ of this sum.

5. Let X be a point on the circle with diameter AB, distinct from A and B. Let t_a,t_b,t_x be the tangents to the circle at A,B,X, respectively. Line AX meets t_b at Z, and line BX meet t_a at Y. Show that the lines YZ,t_x and AB are either concurrent or parallel.

6. An unlimited supply of 8-cent and 15-cent stamps are available. What is the largest amount of postage which cannot be made up exactly?

7. A bus route consists of a circular road of length 10 miles and a straight road of length 1 from a terminus to the point Q on the circular road. It is served by two buses, each of which requires 20 minutes for the round trip. Bus 1, upon leaving the terminus, travels along the straight road, once around the circle and returns along the straight road to the terminus. Bus 2, leaving the terminus 10 minutes after Bus 1, proceeds counterclockwise around the circle. The buses run continuously and wait at stations for a negligible amount of time.

   A man waiting at point P which x miles $(0\le x\le 12)$ from the terminus along the route of Bus 1, wants to travel to the terminus by one of the buses. It is assumed that he chooses the bus which will bring him to his destination at the earliest moment. Let $w(x)$ be the maximum time that his journey (waiting plus travel) can take. Find $w(2)$ and $w(4)$ and sketch the graph of $w(x)$ for $0\le x\le 12$. For what x is $w(x)$ maximal?

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