4th Canadian Mathematical Olympiad 1972

1. Given three distinct unit circles which are tangent to each other, find the radii of the circles which are tangent to all three circles.

2. For non-negative real numbers $a_1, a_2, \dots, a_n$, define $M = a_1a_2 + a_1a_3 + \cdots + a_{n-1}a_n$. Prove that at least one of the numbers $a_1^2, a_2^2, \dots, a_n^2$ does not exceed $\dfrac{2M}{n(n-1)}$.

3. (a) Prove that 10201 is composite in any base greater than 2.

   (b) Prove that 10101 is composite in any base.

4. Construct a quadrilateral $ABCD$ given:

   (i) the lengths of all four sides;

   (ii) that AB and CD are parallel;

   (iii) that BC and DA do not intersect.

5. Prove that the equation $x^3 + 11^3 = y^3$ has no solution in positive integers.

6. Given any distinct real numbers a, b, prove that there exist integers m, n such that $am + bn < 0 < bm + an$.

7. (a) Prove that the values of x for which $x = \dfrac{x^2+1}{198}$ lie between $1/198$ and $197.99\overline{49}$.

   (b) Use the result of (a) to prove that $\sqrt{2} < 1.41\overline{421356}$.

   (c) Is it true that $\sqrt{2} < 1.41421356$?

8. During an election campaign, p different kinds of promises ($p > 0$) are made by various political parties. Any two parties have at least one promise in common, and no two parties have the same set of promises. Prove that there are no more than 2^{p-1} parties.

9. Four distinct lines L_1, L_2, L_3, L_4 are given in the plane so that L_1 and L_2 are respectively parallel to L_3 and L_4. Find the locus of a point moving so that the sum of its distances from the four lines is constant.

10. What is the maximum number of terms in an increasing geometric progression whose all entries come from the set $\{100, 101, \dots, 1000\}$?

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