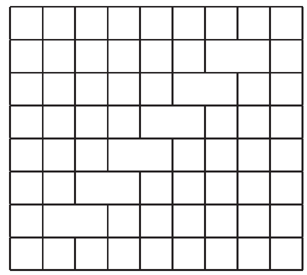
39-th Canadian Mathematical Olympiad 2007

March 28, 2007

1. What is the maximum number of non-overlapping 2×1 dominoes that can be placed on a 8×9 checkerboard if six of them are placed as shown? Each domino must be placed horizontally or vertically so as to cover two adjacent squares of the board.



2. You are given a pair of triangles for which:

   (i) two sides of one triangle are equal in length to two sides of the second triangle, and

   (ii) the triangles are similar, but not necessarily congruent.

   Prove that the ratio of the sides that correspond under the similarity is a number between $\frac{1}{2}(\sqrt{5}-1)$ and $\frac{1}{2}(\sqrt{5}+1)$.

3. Suppose that f is a real-valued function for which

$$f(xy) + f(y-x) \geq f(y+x)$$

   for all real numbers x and y.

   (a) Give a nonconsant polynomial that satisfies the condition.

   (b) Prove that $f(x) \geq 0$ for all real x

4. For two real numbers a, b, with $ab \neq 1$, define operation $\star$ by

$$a \star b = \frac{a+b-2ab}{1-ab}.$$

   Start with a list of $n \geq 2$ real numbers whose entries x all satisfy $0 < x < 1$. Select any two numbers a and b in the list; remove them and put the number $a \star b$ at the end of the list, thereby reducing its length by one. Repeat this procedure until a single number remains.

   (a) Prove that this single number is the same regardless of the choice of pair at each stange.

(b) Suppose that the condition on the numbers x in S is weakened to $0 < x \leq 1$. What happens if S contains exactly one 1?

5. Let the incicrle of triangle ABC touch sides BC, CA, and AB at D, E, and F, respectively. Let Γ, Γ_1, Γ_2, and Γ_3 denote the circumcircles of triangles ABC, AEF, BDF, and CDE respectively. Let Γ and Γ_1 intersect at A and P, Γ and Γ_2 intersect at B and Q, and Γ and Γ_3 intersect at C and R.

   (a) Prove that the circles Γ_1, Γ_2, and Γ_3 intersect in a common point.

   (b) Show that PD, QE, and RF are concurrent.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com