38-th Canadian Mathematical Olympiad 2006

March 29, 2006

1. Let $f(n,k)$ be the number of ways of distributing k candies to n children so that each child receives at most two candies. For example, $f(3,7)=0$, $f(3,6)=1$ and $f(3,4)=6$. Evaluate

$$f(2006,1)+f(2006,4)+f(2006,7)+\cdots+f(2006,1003).$$

2. Let ABC be an acute-angled triangle. Inscribe a rectangle $DEFG$ in this triangle so that D is on AB, E on AC, and F and G on BC. Describe the locus of the intersections of the diagonals of all possible rectangles $DEFG$.

3. In a rectangular array of nonegative real numbers with m rows and n columns, each row and each column contains at least one positive element. Moreover, if a row and a column intersect in a positive element, then the sums of their elements are the same. Prove that $m=n$.

4. Consider a round-robin tournament with $2n+1$ teams, where every two teams play exactly one match and there are no ties. We say that teams X,Y,Z form a *cycle triplet* if X beats Y, Y beats Z, and Z beats X.

   (a) Find the minimum number of cycle triplets possible.

   (b) Find the maximum number of cycle triplets possible.

5. The vertices of a right triangle ABC inscribed in a circle divide the circumference into three arcs. The right angle is at A. To each of the three arcs, we draw a tangent such that its point of tangency is the midpoint of the portion of the tangent intercepted by the lines AB and AC. If the tangency points are D,E and F, show that triangle DEF is equilateral.

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