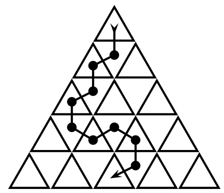
37-th Canadian Mathematical Olympiad 2005

March 30, 2005

1. An equilateral triangle of side length n is divided into unit triangles. Let $f(n)$ be the number of paths from the triangle in the top row to the middle triangle in the bottom row, such that adjacent triangles in a path share a common edge and the path never travels up (from a lower row to a higher row) or revisits a triangle. An example is shown on the picture for $n = 5$. Determine the value of $f(2005)$.



2. Let (a,b,c) be a Pythagorean triple, i.e. a triplet of positive integers with $a^2 + b^2 = c^2$.

   (a) Prove that $(\frac{c}{a} + \frac{c}{b})^2 > 8$.

   (b) Prove that there are no integer n and Pythagorean triple (a,b,c) satisfying $(\frac{c}{a} + \frac{c}{b})^2 = n$.

3. Let S be a set of $n \geq 3$ points in the interior of a circle.

   (a) Show that there are three distinct points $a,b,c \in S$ and three distinct points A,B,C on the circle such that a is (strictly) closer to A than any other point in S, b is closer to B than any other point in S and c is closer to C than any other point in S.

   (b) Show that for no value of n can four such points in S (and corresponding points on the circle) be guaranteed.

4. Let ABC be a triangle with circumradius R, perimeter P and area K. Determine the maximum value of KP/R^3.

5. Let's say that an ordered triple of positive integers (a,b,c) is *n-powerful* if $a \leq b \leq c$, $\gcd(a,b,c) = 1$, and $a^n + b^n + c^n$ is divisible by $a+b+c$. For example, $(1,2,2)$ is 5-powerful.

   (a) Determine all ordered triples (if any) which are n-powerful for all $n \geq 1$.

   (b) Determine all ordered triples (if any) which are 2004-powerful and 2005-powerful, but not 2007-powerful.

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