15-th Baltic Way

Vilnius, Lithuania – November 7, 2004

1. A sequence $a_1, a_2, \dots$ of nonnegative real numbers satisfies for all n
$$a_n + a_{2n} \geq 3n \quad \text{and} \quad a_{n+1} + n \leq 2\sqrt{(n+1)a_n}.$$
   (a) Prove that $a_n \geq n$ holds for every sufficiently large n.
   (b) Give an example of such a sequence.

2. Let $P(x)$ be a polynomial with nonnegative coefficients. Prove that if the inequality $P(x)P(1/x) \geq 1$ holds for $x = 1$ then it holds for every $x > 0$.

3. If p, q, r are positive numbers with $pqr = 1$ and n a natural number, prove that
$$\frac{1}{p^n + q^n + 1} + \frac{1}{q^n + r^n + 1} + \frac{1}{r^n + p^n + 1} \leq 1.$$

4. Let $x_1, x_2, \dots, x_n$ be real numbers with arithmetic mean X. Show that there is an index k such that, for each $i \leq k$, the arithmetic mean of the numbers $x_i, x_{i+1}, \dots, x_k$ does not exceed X.

5. The function f is defined for each integer k by
$$f(k) = (k)_3 + (2k)_5 + (3k)_7 - 6k,$$
where $(k)_{2n+1}$ denotes the multiple of $2n+1$ closest to k. Find the set of values taken by f.

6. A positive integer is written on each of the six faces of a cube. For each vertex of the cube we compute the product of the numbers on the three faces meeting at that vertex. If the sum of these products is 1001, what is the sum of the six numbers on the faces?

7. Find all sets X consisting of at least two positive integers such that for every $m, n \in X$ with $n > m$ there exists $k \in X$ with $n = mk^2$.

8. Prove that for every nonconstant polynomial $f(x)$ with integer coefficients there exists an integer n such that $f(n)$ has at least 2004 distinct prime factors.

9. A set S of $n-1$ natural numbers is given ($n \geq 3$), not all of which are congruent modulo n. Prove that it is possible to choose a non-empty subset of S with the sum of elements divisible by n.

10. Is there an infinite sequence of prime numbers $p_1, p_2, \dots$ satisfying $|p_{n+1} - 2p_n| = 1$ for each $n \in \mathbb{N}$?

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11. Given is an $m \times n$ table in each of whose cells either 1 or -1 is written. Initially there is exactly one -1 in the table. In each step it is allowed to choose a cell containing -1, replace this -1 by 0, and simultaneously change signs of the numbers in the neighbouring (by side) cells. Find all (m,n) for which one can obtain the table containing zeros only using such moves, regardless of the position of the initial -1.

12. There are $2n$ different numbers in a row. By one move we can interchange any two numbers or interchange any three numbers cyclically (i.e. replace a,b,c by b,c,a respectively). What is the minimal number of moves that is always sufficient to arrange the numbers in increasing order?

13. The 25 member states of the EU set up a committee with the following rules:

    (i) The committee should meet daily;

    (ii) At each meeting, at least one member state should be represented;

    (iii) At no two meetings the same set of states is represented;

    (iv) At the n-th meeting, for every $k < n$, the set of states represented should include at least one state that was represented at the k-th meeting.

    At most, for how many days can the committee meet?

14. We say that a *pile* is a set of four or more nuts. Two persons play the following game. They start with one pile of $n \geq 4$ nuts. In a move a player splits one of the available piles into two nonempty sets (which are not necessrily piles). A player who cannot make a move loses. For which values of n does the first player have a winning strategy?

15. A circle is divided into 13 segments, numbered consecutively from 1 to 13. Five fleas called A,B,C,D and E are sitting at the segments $1,2,3,4,5$ respectively. A flea is allowed to jump to an empty segment five positions away in either direction around the circle. Only one flea jumps at a time. After several jumps, the fleas are back in the segments 1,2,3,4,5, but possibly in some other order. Which orders are possible?

16. A secant through point P exterior to a given circle intersects the circle at A and B, and a tangent touches the circle at C on the same side of the diameter through P as A and B. The projection of point C on this diameter is Q. Prove that the line QC bisects the angle $\angle AQB$.

17. A quadrilateral with side lengths x,y,z,u is inscribed in a rectangle with sides 3 and 4, with one vertex on each side of the rectangle. Prove that

$$25 \leq x^2 + y^2 + z^2 + u^2 \leq 50.$$

18. A ray through the vertex A of a triangle ABC intersects the side BC at X and the circumcircle of $\triangle ABC$ at Y. Prove that

$$\frac{1}{AX} + \frac{1}{XY} \geq \frac{4}{BC}.$$

19. In a triangle ABC, D is the midpoint of BC and M the point on side BC such that $\angle BAM = \angle DAC$. The circumcircles of triangles CAM and BAM meet the sides AB and AC again at points L and K, respectively. Show that $KL \parallel BC$.

20. Three circular arcs $\omega_1, \omega_2, \omega_3$ with common endpoints A and B lie on the same side of the line AB with ω_2 between ω_1 and ω_3. Two rays emanating from B intersect these arcs at M_1, M_2, M_3 and K_1, K_2, K_3, respectively. Prove that

$$\frac{M_1M_2}{M_2M_3} = \frac{K_1K_2}{K_2K_3}.$$

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