13-th Baltic Way

Tartu – November 2, 2002

1. Solve in the real numbers the system of equations

$$\begin{aligned} a^3+3ab^2+3ac^2-6abc &= 1 \\ b^3+3ba^2+3bc^2-6abc &= 1 \\ c^3+3ca^2+3cb^2-6abc &= 1. \end{aligned}$$

2. Real numbers a, b, c, d satisfy

$$a+b+c+d=-2 \quad \text{and} \quad ab+ac+ad+bc+bd+cd=0.$$

   Prove that at least one of the numbers a, b, c, d is not greater than -1.

3. Find all sequences $(a_n)_{n=0}^{\infty}$ of real numbers which satisfy

$$a_{m^2+n^2}=a_m^2+a_n^2 \quad \text{for all integers } m,n \geq 0.$$

4. Let $x_1, x_2, \dots, x_n$ be nonnegative real numbers with $x_1+\cdots+x_n=1$. Prove that

$$\sum_{i=1}^{n} x_i(1-x_i)^2 \leq \left(1-\frac{1}{n}\right)^2.$$

5. Find all pairs (a,b) of positive rational numbers such that

$$\sqrt{a}+\sqrt{b}=\sqrt{2+\sqrt{3}}.$$

6. The following solitaire game is played on an $m \times n$ rectangular board ($m,n \geq 2$) divided into unit squares, with a rook placed on some square. At each move, the rook can be moved an arbitrary number of squares horizontally or vertically, with the extra condition that each move has to be made in the 90° clockwise direction compared to the previous one. For which values of m and n is it possible that the rook visits every square of the board exactly once and returns to the first square? (The rook is considered to visit only those squares it stops on, and not the ones it steps over.)

7. The plane is divided into regions by n convex quadrilaterals (one of the regions is infinite). Find the maximum possible number of these regions.

8. Let P be a set of $n \geq 3$ points in the plane, no three of which are on a line. How many possibilities are there to choose a set T of $\binom{n-1}{2}$ triangles with vertices in P, such that each triangle in T has a side that is not a side of any other triangle in T?

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9. Two magicians show the following trick. The first magician goes out of the room. The second magician takes a deck of 100 cards labelled by numbers $1,2,\dots,100$ and asks three spectators to choose in turn one card each. The second magician sees what card each spectator has taken. Then he adds one more card from the rest of the deck. Spectators shuffle these four cards, call the first magician and give him these four cards. The first magician looks at the four cards and guesses what card was chosen by each of the spectators. Prove that the magicians can perform this trick.

10. Let N be a positive integer. Two persons play the following game. The first player writes a list of positive integers not greater than 25, not necessarily different, with the sum at least 200. The second player wins if he can select some of these numbers so that their sum S satisfies the condition $|S-200| \le N$. What is the smallest value of N for which the second player has a winning strategy?

11. Consider n points in the plane such that no three of them are collinear and no two of the distances between them are equal. One by one, we simultaneously connect each point to the two points nearest to it by segments. Prove that there is no point which is connected by segments to more than 11 points.

12. A set S of four distinct points is given in the plane. Suppose that for any point $X \in S$ the remaining points can be denoted by Y, Z and W so that $XY = XZ + XW$. Prove that all the four points lie on a line.

13. Let ABC be an acute triangle with $\angle A > \angle C$, and let D be a point on side AC such that $AB = BD$. Let F be a point on the smaller arc AC or the circumcircle of triangle ABC such that FD is perpendicular to BC. Prove that $FB \perp AC$.

14. Let L, M and N be points on sides AC, AB and BC of triangle ABC, respectively, such that BL is the bisector of angle ABC and segments AN, BL and CM have a common point. Prove that if $\angle ALB = \angle MNB$ then $\angle LNM = 90^\circ$.

15. A spider and a fly are sitting on a cube. The fly wants to maximize the length of the shortest path to the spider along the surface of the cube. Is it necessarily best for the fly to be at the point symmetric to the spider with respect to the center of the cube?

16. Find all nonnegative integers m for which $a_m = \left(2^{2m+1}\right)^2 + 1$ is divisible by at most two different primes.

17. Show that the sequence $\binom{2002}{2002}, \binom{2003}{2002}, \binom{2004}{2002}, \dots$ is periodic modulo 2002.

18. Find all integers $n > 1$ such that any prime divisor of $n^6 - 1$ is a divisor of $(n^3 - 1)(n^2 - 1)$.

19. Let n be a positive integer. Prove that the equation

$$x + y + \frac{1}{x} + \frac{1}{y} = 3n$$

has no solutions in positive rational numbers.

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20. Does there exist an infinite non-constant arithmetic progression, each term of which is of the form a^b, where a and b are positive integers with $b \geq 2$?

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