12-th Baltic Way

Hamburg, Germany – November 1, 2001

1. Eight problems were prepared for an examination. Each student was given three of them, but no two students received more than one common problem. What is the largest possible number of students?

2. Let $n \geq 2$ be an integer. Find whether there exist n pairwise nonintersecting nonempty subsets of $\mathbb{N}$ such that each positive integer can be uniquely expressed as a sum of at most n integers, all from different subsets.

3. The numbers $1, 2, \dots, 49$ are placed in a 7×7 array, and the sum of the numbers in each row and in each column is computed. Some of these 14 sums are odd while others are even. Let A denote the sum of all the odd sums and B the sum of all even sums. Is it possible that the numbers were placed in the array in such a way that $A = B$?

4. Let p and q be two different primes. Prove that
$$\left[\frac{p}{q}\right] + \left[\frac{2p}{q}\right] + \cdots + \left[\frac{(q-1)p}{q}\right] = \frac{1}{2}(p-1)(q-1).$$

5. Let 2001 given points on a circle be colored either red or green. In one step all points are recolored simultaneously in the following way: If both direct neighbors of a point P have the same color as P, then the color of P remains unchanged, otherwise the color of P is changed. Starting with the first coloring F_1 and we obtain the colorings $F_2, F_3, \dots$ after several recoloring steps. Prove that there is an $n_0 \leq 1000$ such that $F_{n_0} = F_{n_0+2}$. Is the assertion necessarily true if 1000 is replaced by 999?

6. Points A, B, C, D, E lie on a circle c in this order and satisfy $AB \parallel EC$ and $AC \parallel ED$. The line tangent to the circle c at E meets the line AB at P. The lines BD and EC meet at Q. Prove that $AC = PQ$.

7. A circle passing through the vertex A of a parallelogram $ABCD$ meets segments AB, AC, and AD at points M, K, N respectively. Prove that
$$AB \cdot AM + AD \cdot AN = AK \cdot AC.$$

8. Let N be the midpoint of side BC of a convex quadrilateral $ABCD$. Suppose that $\angle AND = 135^\circ$. Prove that $AB + CD + \frac{1}{\sqrt{2}}BC \geq AD$.

9. Given a rhombus $ABCD$, find the locus of the points P lying inside the rhombus and satisfying $\angle APD + \angle BPC = 180^\circ$.

10. In a triangle ABC, the bisector of $\angle BAC$ meets the side BC at D. Knowing that $BD \cdot CD = AD^2$ and $\angle ADB = 45^\circ$, determine the angles of $\triangle ABC$.

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11. A function $f : \mathbb{N} \to \mathbb{R}$ satisfies the condition

$$f(ab) = f(d)\left(f\left(\frac{a}{d}\right) + f\left(\frac{b}{d}\right)\right) \quad \text{for all } a, b \in \mathbb{N},$$

where $d = \gcd(a, b)$. Determine all possible values of $f(2001)$.

12. Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that

$$\sum_{i=1}^{n} a_i^3 = 3 \quad \text{and} \quad \sum_{i=1}^{n} a_i^5 = 5.$$

Prove that $\sum_{i=1}^{n} a_i > \frac{3}{2}$.

13. A sequence $a_0, a_1, a_2, \ldots$ satisfies $a_0 = 1$ and $a_n = a_{[7n/9]} + a_{[n/9]}$ for $n \geq 1$. Prove that there exists a positive integer k with $a_k < \frac{k}{2001!}$.

14. On each of the $2n$ cards there is a real number x with $1 \leq x \leq 2$. Prove that the cards can be divided into two heaps with the sums of corresponding numbers s_1 and s_2 so that $\frac{n}{n+1} \leq \frac{s_1}{s_2} \leq 1$.

15. The sequence $(a_n)_{n=0}^{\infty}$ of positive real numbers has the property that

$$ia_i^2 \geq (i+1)a_{i-1}a_{i+1} \quad \text{for } i = 1, 2, \ldots.$$

For some $x, y > 0$, denote $b_i = xa_i + ya_{i-1}$ for $i = 1, 2, \ldots$. Prove that for all $i \geq 2$ we have $ib_i^2 > (i+1)b_{i-1}b_{i+1}$.

16. A function $f : \mathbb{N} \to \mathbb{R}$ is such that for all $n > 1$ there exists a prime divisor p of n such that $f(n) = f(\frac{n}{p}) - f(p)$. Given that $f(2001) = 1$, what is the value of $f(2002)$?

17. Let n be a positive integer. Prove that one can choose no less than $2^{n-1} + n$ numbers from the set $\{1, 2, \ldots, 2^n\}$ such that for any two different chosen numbers x, y, $x + y$ does not divide xy.

18. Let a be an odd integer. Prove that $a^{2^n} + 2^{2^n}$ and $a^{2^m} + 2^{2^m}$ are coprime for all positive integers $n \neq m$.

19. What is the smallest positive odd integer having the same number of positive divisors as 360?

20. From a quadruple of integers (a, b, c, d) each of the sequences

$$(c, d, a, b), \quad (b, a, d, c), \quad (a + nc, b + nd, c, d), \quad (a + nb, b, c + nd, d)$$

for an arbitrary integer n can be obtained by one step. Is it possible to obtain $(3, 4, 5, 7)$ from $(1, 2, 3, 4)$ through a sequence of such steps?

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