11-th Baltic Way

Oslo, Norway – November 5, 2000

1. Let K be a point inside a triangle ABC. Let M and N be points such that M and K are on opposite sides of the line AB, and N and K are on opposite sides of the line BC. Assume that $\angle MAB = \angle MBA = \angle NBC = \angle NCB = \angle KAC = \angle KCA$. Show that $MBNK$ is a parallelogram.

2. Given an isosceles triangle ABC with $\angle A = 90^\circ$, let M be the midpoint of AB. The line passing through A and perpendicular to CM intersects the side BC at P. Prove that $\angle AMC = \angle BMP$.

3. Given a triangle ABC with $\angle A = 90^\circ$ and $AB \neq AC$. The points D, E, F lie on the sides BC, CA, AB respectively in such a way that $AFDE$ is a square. Prove that the line BC, the line FE and the tangent to the circumcircle of $\triangle ABC$ at A intersect in one point.

4. Points K and L lie on the sides AB and AC respectively of a triangle ABC with $\angle A = 120^\circ$. Equilateral triangles BKP and CLQ are constructed outside $\triangle ABC$. Prove that $PQ \geq \frac{\sqrt{3}}{2}(AB + AC)$.

5. Let ABC be a triangle such that $\frac{BC}{AB - BC} = \frac{AB + BC}{AC}$. Determine the ratio $\angle A : \angle C$.

6. Fredek runs a private hotel. He claims that whenever $n \geq 3$ guests visit the hotel, it is possible to select two guests that have equally many acquaintances among the other guests, and that also have a common acquaintance or a common unknown among the guests. For which values of n is Fredek right? (Acquaintance is a symmetric relation.)

7. In a 40×50 array of control buttons, each button has two states: on and off. By touching a button, its state and the states of all buttons in the same row and in the same column are switched. Prove that the array of control buttons can be altered from the all-off state to the all-on state by touching buttons successively, and determine the least number of touches needed to do so.

8. Fourteen friends met at a party. One of them, Fredek, wanted to go to bed early. He said goodbye to 10 of his friends, forgot about the remaining 3, and went to bed. After a while he returned to the party, said goodbye to 10 of his friends (not necessarily the same as before), and went to bed. Later Fredek came back a number of times, each time saying goodbye to exactly 10 of his friends, and then went back to bed. As soon as he had said goodbye to each of his friends at least once, he did not come back again. In the morning Fredek realized that he had said goodbye a different number of times to each of his 13 friends. What is the smallest possible number of times that Fredek returned to the party?

9. There is a frog jumping on a $2k \times 2k$ chessboard composed of unit squares. The frog's jumps are $\sqrt{1 + k^2}$ long, taking the frog from the center of a square to

the center of another square. Some m squares are marked with a $\times$, and all the squares onto which the frog can jump from square with a $\times$ (whether they carry an $\times$ or not) are marked with an $\circ$. There are n squares with a $\circ$. Prove that $n \geq m$.

10. Two positive integers are written on the blackboard. Initially, one of them is 2000 and the other is smaller than 2000. If the arithmetic mean m of the two numbers on the blackboard is an integer, one of the two numbers is erased and replaced by m. Prove that this operation cannot be performed more than ten times. Give an example where the operation is performed ten times.

11. A sequence of positive integers $a_1, a_2, \dots$ is such that for each m and n the following holds: if m divides n and $m < n$, then a_m divides a_n and $a_m < a_n$. Find the least possible value of a_{2000}.

12. Let $x_1, x_2, \dots, x_n$ be positive integers such that no one of them is an initial fragment of any other (for example, 12 is an initial fragment of $\underline{12}$, $\underline{12}5$ and $\underline{12}405$). Prove that
$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} < 3.$$

13. Let $a_1, a_2, \dots, a_n$ be an arithmetic progression of integers such that $i \mid a_i$ for $i = 1, 2, \dots, n-1$ and $n \nmid a_n$. Prove that n is a prime power.

14. Find all positive integers n having exactly $n/100$ positive divisors.

15. Let n be a positive integer not divisible by 2 or 3. Prove that for all integers k, the number $(k+1)^n - k^n - 1$ is divisible by $k^2 + k + 1$.

16. Prove that for all positive real numbers a, b, c
$$\sqrt{a^2 - ab + b^2} + \sqrt{b^2 - bc + c^2} \geq \sqrt{a^2 + ac + c^2}.$$

17. Find all real solutions to the following system of equations:
$$\begin{cases} x + y + z + t = 5 \\ xy + yz + zt + tx = 4 \\ xyz + yzt + ztx + txy = 3 \\ xyzt = -1. \end{cases}$$

18. Determine all positive real numbers x and y satisfying the equation
$$x + y + \frac{1}{x} + \frac{1}{y} + 4 = 2 \cdot (\sqrt{2x+1} + \sqrt{2y+1}).$$

19. Let $t \geq \frac{1}{2}$ be a real number and n a positive integer. Prove that
$$t^{2n} \geq (t-1)^{2n} + (2t-1)^n.$$

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20. For every positive integer n, let

$$x_n = \frac{(2n+1)(2n+3)\cdots(4n-1)(4n+1)}{(2n)(2n+2)\cdots(4n-2)(4n)}.$$

Prove that $\frac{1}{4n} < x_n - \sqrt{2} < \frac{2}{n}$.

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