9-th Baltic Way

Warsaw, Poland – November 8, 1998

1. Find all functions $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ of two variables satisfying for all x, y:
$$f(x,x) = x, \quad f(x,y) = f(y,x), \quad (x+y)f(x,y) = yf(x,x+y).$$

2. A triple (a,b,c) of positive integers is called *quasi-Pythagorean* if a,b,c are the sides of a triangle with an angle of 120° corresponding to side c. Prove that if (a,b,c) is a quazi-Pythagorean triple then c has a prime divisor greater than 5.

3. Find all integral solutions (x,y) to the equation $2x^2 + 5y^2 = 11(xy - 11)$.

4. Let P be a polynomial with integer coefficients. Suppose that $P(n)$ is a three-digit natural number for $n = 1, 2, \dots, 1998$. Prove that the polynomial P has no integer zeros.

5. Given an odd digit a and an even digit b, prove that for each $n \in \mathbb{N}$ there is a multiple of 2^n whose decimal expansion consist only of digits a and b.

6. A polynomial P of degree 6 and real numbers a, b with $0 < a < b$ satisfy $P(a) = P(-a)$, $P(b) = P(-b)$ and $P'(0) = 0$. Prove that $P(x) = P(-x)$ for all real x.

7. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) + f(y) = f(f(x)f(y))$ for all real x, y.

8. Denote $P_k(x) = 1 + x + x^2 + \cdots + x^{k-1}$. Show that for all x and $n \in \mathbb{N}$
$$\sum_{k=1}^{n} \binom{n}{k} P_k(x) = 2^{n-1} P_n\left(\frac{1+x}{2}\right).$$

9. Given numbers $0 < \alpha < \beta < \pi/2$, let $0 < \gamma, \delta < \pi/2$ be the numbers such that $\tan\gamma$ is the arithmetic mean of $\tan\alpha$ and $\tan\beta$ and $\sec\delta$ is the arithmetic mean of $\sec\alpha$ and $\sec\beta$. Prove that $\gamma < \delta$.

10. A regular n-gon and a regular $(n-1)$-gon are inscribed in a unit circleFor each vertex of the n-gon consider the distance to the nearest vertex of the $(n-1)$-gon measured along the circumference. Let S be the sum of these n distances. Prove that S depends only on n and not on the relative position of the two polygons.

11. If a, b, c are the sides and R the circumradius of a triangle ABC, prove that
$$R \geq \frac{a^2 + b^2}{2\sqrt{2a^2 + 2b^2 - c^2}}.$$
When does equality hold?

12. In a triangle ABC with a right angle at A, point D on the side BC satisfies $\angle BDA = 2\angle BAD$. Prove that
$$\frac{1}{AD} = \frac{1}{2}\left(\frac{1}{BD} + \frac{1}{CD}\right).$$

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13. In a convex pentagon $ABCDE$ with AE parallel to BC and $\angle ADE = \angle BDC$, the diagonals AC and BE intersect at P. Prove that $\angle EAD = \angle BDP$ and $\angle CBD = \angle ADP$.

14. In a triangle ABC with $AB < AC$, the line through B parallel to AC meets the external bisector of $\angle BAC$ at D and the line through C parallel to AB meets this bisector at E. Point F on side AC satisfies $FC = AB$. Prove that $DF = FE$.

15. Let D be a point on the altitude from A of an acute triangle ABC and let E be the point on this altitude satisfying $AE/ED = CD/DB$. If F is the projection of D on BE, prove that $\angle AFC = 90^\circ$.

16. Can one put 42 pieces of size 4×1 on a 13×13 square board so that only the central field of the board remains uncovered?

17. There are nk objects and k boxes, each of which can hold n objects. Each object is colored with one of k colors. Show that the objects can be packed in the boxes so that each box contains objects of at most two colors.

18. Find all $n \in \mathbb{N}$ for which there exists a set S with the following properties:

    (i) S consists of n positive integers smaller than 2^{n-1};

    (ii) Any two distinct subsets of S have different sums of elements.

19. Two teams of 1000 players each played a ping-pong match in which every two players from different teams played exactly one match (no draws). Prove that there exist ten players from the same team such that every member of the other team lost to at least one of those ten players.

20. A natural number m is said to *cover* the number 1998 if the digits $1, 9, 9, 8$ occur in this order as digits of m. (For instance, 1998 is covered by 215993698 but not by 213326798.) Let $k(n)$ be the number of n-digit positive numbers $(n \geq 5)$ that cover 1998 and whose all digits are nonzero. What is the remainder of $k(n)$ when divided by 8.