6-th Baltic Way

Västerås, Sweden – November 12, 1995

1. Find all triples (x,y,z) of positive integers satisfying the system

$$\begin{cases} x^2 = 2(y+z) \\ x^6 = y^6 + z^6 + 31(y^2 + z^2) \end{cases}$$

2. Let a and k be positive integers such that $a^2 + k$ divides $(a-1)a(a+1)$. Prove that $k \geq a$.

3. The positive integers a, b, c, with a and c odd, are pairwise relatively prime and satisfy $a^2 + b^2 = c^2$. Prove that $b + c$ is a perfect square.

4. John is older than Mary. He notices that if he switches the two digits of his age (an integer), he gets Mary's age. Moreover, the difference between the squares of their ages is a square of an integer. How old are Mary and John?

5. Let $a < b < c$ be three positive integers. Prove that among any $2c$ consecutive positive integers there exist three different numbers x, y, z such that abc divides xyz.

6. Prove that for positive a, b, c, d

$$\frac{a+c}{a+b} + \frac{b+d}{b+c} + \frac{c+a}{c+d} + \frac{d+b}{d+a} \geq 4$$

7. Prove that $\sin^3 18^\circ + \sin^2 18^\circ = \frac{1}{8}$.

8. Real numbers a, b and c satisfy the inequalities $|a| \geq |b+c|$, $|b| \geq |c+a|$ and $|c| \geq |a+b|$. Prove that $a + b + c = 0$.

9. Prove that

$$\frac{1995}{2} - \frac{1994}{3} + \frac{1993}{4} - \cdots - \frac{2}{1995} + \frac{1}{1996} = \frac{1}{999} + \frac{3}{1000} + \cdots + \frac{1995}{1996}.$$

10. Find all real-valued functions f defined on the set of all nonzero real numbers such that:

    (i) $f(1) = 1$,

    (ii) $f\left(\frac{1}{x+y}\right) = f\left(\frac{1}{x}\right) + f\left(\frac{1}{y}\right)$ for all non-zero x, y, $x+y$,

    (iii) $(x+y) \cdot f(x+y) = xy \cdot f(x) \cdot f(y)$ for all nonzero x, y, $x+y$.

11. In how many ways can the set of integers $\{1, 2, \ldots, 1995\}$ be partitioned into three nonempty sets so that none of these sets contains a pair of consecutive integers?

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
Typed in LaTeX by Ercole Suppa
www.imomath.com

12. Assume we have 95 boxes and 19 balls distributed in these boxes in an arbitrary manner. We take 6 new balls at a time and place them in 6 of the boxes, one ball in each of the six. Can we, by repeating this process a suitable number of times, achieve a situation in which each of the 95 boxes contains an equal number of balls?

13. Consider the following two-person game. A number of pebbles are situated on the table. Two players make their moves alternately. A move consists of taking off the table x pebbles where x is the square of a positive integer. The player who is unable to make a move loses. Prove that there are infinitely many initial situations in which the second player can win no matter how his opponent plays.

14. There are n fleas on an infinite sheet of triangulated paper. Initially the fleas are in different small triangles, all of which are inside some equilateral triangle consisting of n^2 small triangles. Once a second each flea jumps from its original triangle to one of the three small triangles having a common vertex but no common side with it. For which natural numbers n does there exist an initial configuration such that after a finite number of jumps all the n fleas can meet in a single small triangle?

15. A polygon with $2n+1$ vertices is given. Show that it is possible to assign numbers $1,2,\ldots,4n+2$ to the vertices and midpoints of the sides of the polygon so that for each side the sum of the three numbers assigned to it is the same.

16. In the triangle ABC, ℓ is the bisector of the external angle at C. The line through the midpoint O of AB parallel to ℓ meets AC at E. Determine $|CE|$ if $|AC|=7$ and $|CB|=4$.

17. Show that there is a number α such that for any triangle ABC the inequality

$$\max(h_A,h_B,h_C)\leq\alpha\cdot\min(m_A,m_B,m_C)$$

holds, where h_A,h_B,h_C are the lengths of the altitudes and m_A,m_B,m_C the lengths of the medians. Find the smallest possible value of α.

18. Let M be the midpoint of the side AC of a triangle ABC and let H be the foot of the altitude from B. Let P and Q be orthogonal projections of A and C on the bisector of the angle B. Prove that the points H,P,M and Q lie on a circle.

19. The following construction is used for training astronauts:
A circle C_2 of radius $2R$ rolls along the inside of another, fixed circle C_1 of radius nR, where n is an integer greater than 2. The astronaut is fastened to a third circle C_3 of radius R which rolls along the inside of circle C_2 in such a way that the touching point of the circles C_2 and C_3 remains at maximum distance from the touching point of the circles C_1 and C_2 at all times.
How many revolutions (relative to the ground) does the astronaut perform together with the circle C_3 while the circle C_2 completes one full lap around the inside of circle C_1?

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
Typed in LaTeX by Ercole Suppa
www.imomath.com

20. Prove that if all vertices of a convex pentagon have integral coordinates then the area of this pentagon is not less than $\frac{5}{2}$.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
Typed in LaTeX by Ercole Suppa
www.imomath.com