4-th Baltic Way

Tartu, Estonia – November 11, 1994

1. Let us define $a \circ b = a + b - ab$. Find all triples (x, y, z) of integers satisfying
$$(x \circ y) \circ z + (y \circ z) \circ x + (z \circ x) \circ y = 0.$$

2. Let $a_1, a_2, \dots, a_9$ be non-negative numbers with $a_1 = a_9 = 0$ and at least one of the numbers is non-zero. Prove that for some i the inequality $a_{i-1} + a_{i+1} < 2a_i$ holds. Does the statement still hold if the constant 2 is replaced to 1.9?

3. Find the largest value of the expression
$$xy + x\sqrt{1-y^2} + y\sqrt{1-x^2} - \sqrt{(1-x^2)(1-y^2)}.$$

4. Is there an integer n for which $\sqrt{n-1} + \sqrt{n+1}$ is rational?

5. Let $p(x)$ be a polynomial with integer coefficients such that both equations $p(x) = 1$ and $p(x) = 3$ have integer solutions. Can the equation $p(x) = 2$ have two different integer solutions?

6. Prove that any fraction $\frac{p}{q}$, where p and q are coprime positive integers and q is odd, can be written as $\frac{n}{2^k - 1}$ for some positive integers n, k.

7. Let $p > 2$ be a prime number and m, n be coprime integers such that
$$1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{(p-1)^3} = \frac{m}{n}.$$
Show that m is divisible by p.

8. Show that for any integer $a \geq 5$ there exist integers b and c with $c \geq b \geq a$ such that a, b, c are the side lengths of a right-angled triangle.

9. Find all pairs of positive integers (a, b) for which $2^a + 3^b$ is a square.

10. Find the number of positive integers that satisfy the following conditions:
    (a) All of its digits are from the set $\{1, 2, 3, 4, 5\}$;
    (b) Any two consecutive digits differ by 1;
    (c) It has exactly 1994 digits?

11. Let NS and EW be two perpendicular diameters of a circle $\mathscr{C}$. A line l touches $\mathscr{C}$ at point S. Let A and B be two points on $\mathscr{C}$, symmetric with respect to EW. Lines NA and NB meet l at A' and B', respectively. Show that $SA' \cdot SB' = SN^2$.

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12. A circle touches the sides A_2A_3, A_3A_1 and A_1A_2 of a triangle $A_1A_2A_3$ at points S_1, S_2, S_3 respectively. Let O_1, O_2, O_3 be the incenters of triangles $A_1S_2S_3$, $A_2S_3S_1$ and $A_3S_1S_2$, respectively. Prove that the lines O_1S_1, O_2S_2, and O_3S_3 are concurrent.

13. Find the smallest side length a of a square within which one can place five non-intersecting disks of radius 1.

14. Let α, β, γ be the angles of a triangle opposite to its sides a, b, c, respectively. Prove the inequality

$$a\left(\frac{1}{\beta}+\frac{1}{\gamma}\right)+b\left(\frac{1}{\gamma}+\frac{1}{\alpha}\right)+c\left(\frac{1}{\alpha}+\frac{1}{\beta}\right) \geq 2\left(\frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma}\right).$$

15. Is there a triangle whose all sides and altitudes have integer lengths and whose perimeter equals 1995?

16. The Wonder Island is inhabited by hedgehogs consisting of three unit segments with a common endpoint and forming the angles of 120° with each other. Assuming that the hedgehogs lie flat on the island and no two touch each other, prove that there are finitely many of them on Wonder island.

17. In a certain kingdom, the king has decided to build 25 new towns on 13 uninhabited islands, at least one town on each island. Direct ferry connections will be established between any two new towns which are on different islands. Determine the least possible number of these connections.

18. There are $n > 2$ lines in the plane, no two of which are parallel and no three concurrent. Every intersection point of these lines is labelled with a natural number between 1 and $n-1$. Prove that these labels can be assigned in such a way that every line contains all the labels from 1 to $n-1$ if and only if n is even.

19. The Wonder Island Intelligence Service has 16 spies in Tartu. Each spy watches on some of his colleagues. It is known that if spy A watches on spy B, then B does not watch on A. Moreover, any 10 spies can be numbered in such a way that the first spy watches on the second, the second watches on the third, etc, the tenth watches on the first. Prove that any 11 spies can also be numbered in a similar manner.

20. An equilateral triangle is divided into 3000^2 congruent equilateral triangles. Each vertex of the small triangles is painted in in one of three colors. Show that there exists a triangle whose sides are parallel to the sides of the original triangle and whose vertices are all of the same color.