# 4-th Baltic Way

## Riga, Latvia – November 13, 1993

1. Suppose that the three-digit decimal numbers $\overline{a_1a_2a_3}$ and $\overline{a_3a_2a_1}$, with $a_1$ and $a_3$ different and nonzero, have the property that their squares are five-digit numbers of the form $\overline{b_1b_2b_3b_4b_5}$ and $\overline{b_5b_4b_3b_2b_1}$. Find all such three-digit numbers.

2. Do there exist integers $a > b > 1$ such that for each positive integer $k$ there is a positive integer $n$ for which $an+b$ is a $k$-th power of an integer?

3. A positive integer will be called "interesting" if it is a product of two (not necessarily distinct) prime numbers. What is the greatest number of consecutive positive integers all of which are "interesting"?

4. Find all integers $n$ for which
$$\sqrt{\frac{25}{2}+\sqrt{\frac{625}{4}-n}}+\sqrt{\frac{25}{2}-\sqrt{\frac{625}{4}-n}}$$
is an integer.

5. Prove that $n^{12}-n^8-n^4+1$ is divisible by $2^9$ for any odd positive integer $n$.

6. Two functions $f, g$ from the interval $(2,4)$ into itself satisfy $f(g(x)) = g(f(x)) = x$ and $f(x)\cdot g(x) = x^2$ for all $x$. Prove that $f(3) = g(3)$.

7. Find all integral solutions of the system
$$\begin{cases} z^x = y^{2x} \\ 2^z = 4^x \\ x+y+z = 20. \end{cases}$$

8. Compute the sum of all positive integers whose digits form a strictly monotone sequence.

9. Solve the system of equations:
$$\begin{cases} x^5 &= y+y^5 \\ y^5 &= z+z^5 \\ z^5 &= t+t^5 \\ t^5 &= x+x^5. \end{cases}$$

10. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be two finite sequences of real numbers, and let $a_1', \ldots, a_n'$ and $b_1', \ldots, b_n'$ be the rearrangements of these sequences in the increasing order. Prove that
$$\max_{1\le i\le n} |a_i - b_i| \ge \max_{1\le i\le n} |a_i' - b_i'|.$$



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11. An equilateral triangle is divided into $n^2$ congruent equilateral triangles. A spider stands at one of the vertices, a fly at another. Alternately each of them moves to a neighbouring vertex. Show that the spider can always catch the fly.

12. There are 13 cities in a certain kingdom. Some pairs of cities are connected by a two-way direct bus, train or plane route. What is the least possible number of routes, if for any two means of transportation one can go from any city to any other using routes of these two types only?

13. An equilateral triangle is divided into 100 congruent equilateral triangles. What is the greatest number of vertices of small triangles that can be chosen so that no two of them lie on a line parallel to a side of the large triangle?

14. A square is divided into 16 equal squares, obtaining 25 different vertices. What is the least number of vertices that must be removed, so that no four remaining vertices are the vertices of a square with sides parallel to the sides of the initial square?

15. On each face of two dice some positive integer is written. The two dice are thrown and the numbers on the top faces are added. Can one select the integers on the faces so that the possible sums are $2,3,4,\dots,13$, all equally likely?

16. Two disjoint circles of radius $r$ are drawn in the plane. A line meets the first circle at points $A$, $B$ and the other at $C$, $D$ so that $AB = BC = CD = 14$. Another line intersects the circles at points $E$, $F$ and $G$, $H$ respectively, so that $EF = FG = GH = 6$. Find the radius $r$.

17. Three pairwise non-parallel lines are given in the plane. Three points are moving along these lines with different non-zero velocities, one on each line. Is it possible that at no moment are the three points collinear?

18. In triangle $ABC$ with $AB = 15$, $BC = 12$ and $CA = 13$, the median $AM$ and bisector $BK$ intersect at point $O$, where $M$ and $K$ are on $BC$ and $AC$ respectively. Prove that if $L$ is the orthogonal projection of $O$ onto $AB$, then $\angle OLK = \angle OLM$.

19. A convex quadrilateral $ABCD$ is inscribed in a circle with the center $O$. The angles $\angle AOB$, $\angle BOC$, $\angle COD$ and $\angle DOA$, taken in some order, are equal to the angles of the quadrilateral $ABCD$. Show that $ABCD$ is a square.

20. A tetrahedron is said to be *good* if all its edges are equal and all its vertices lie on the boundary of a unit cube. Find all possible volumes of a good tetrahedron.



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