45-th Bulgarian Mathematical Olympiad 1996

Fourth Round

First Day

1. Find all prime numbers p and q such that $\dfrac{(5^p-2^p)(5^q-2^q)}{pq}$ is an integer.

2. Find the side length of the smallest equilateral triangle in which three disks with radii $2,3,4$ and with disjoint interior points can be placed.

3. The quadratic polynomials $f(x)$ and $g(x)$ with real coefficients have the following property: If $g(x)$ is an integer for some $x>0$, then $f(x)$ is also an integer. Prove that there are integers m,n such that $f(x)=mg(x)+n$ for all real x.

Second Day

4. The sequence (a_n) is defined by $a_1=1$ and $a_{n+1}=\dfrac{a_n}{n}+\dfrac{n}{a_n}$ for all $n\geq 1$. Prove that $[a_n^2]=n$ for all $n\geq 4$.

5. Let $ABCD$ be a cyclic quadrilateral. The lines AB and CD meet at E, and the diagonals AC and BD meet at F. The circumcircles of the triangles AFD and BFC intersect again at $H\neq F$. Prove that $\angle EHF=90^\circ$.

6. A square table 7×7 with the four corner squares deleted is given.

   (a) What is the smallest number of squares that need to be colored black so that every 5-square Greek cross (i.e. a square 3×3 with the four corner unit squares cut off) contains at least one black square?

   (b) Prove that it is possible to write integers in each square of the table in such a way that the sum of the integers in each Greek cross is negative while the sum of all integers in the table is positive.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com