43-th Bulgarian Mathematical Olympiad 1994

Fourth Round

First Day

1. Two circles $k_1(O_1,R)$ and $k_2(O_2,r)$ are given in the plane such that $R \geq \sqrt{2}r$ and $O_1O_2 = \sqrt{R^2+r^2-r\sqrt{4R^2+r^2}}$. Let A be an arbitrary point on k_1. The tangents from A to k_2 touch k_2 at B and C and intersect k_1 again at D and E, respectively. Prove that $BD \cdot CE = r^2$.

2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$xf(x) - yf(y) = (x-y)f(x+y) \quad \text{for all } x,y \in \mathbb{R}.$$

3. Let p be a prime number. Determine all positive integers x,y,z such that
$$x^p + y^p = p^z.$$

Second Day

4. Let I be the incenter of a non-isosceles triangle ABC, and let the incircle touch BC,CA,AB at A',B',C' respectively. Prove that the circumcenters of triangles IAA',IBB',ICC' are collinear.

5. Let k be a positive integer and r_n be the remainder when $\binom{2n}{n}$ is divided by k. Find all k for which the sequence $(r_n)_{n=1}^{\infty}$ is eventually periodic.

6. Let n be a positive integer and $\mathscr{A}$ be a family of subsets of the set $\{1,2,\dots,n\}$, none of which contains another subset from $\mathscr{A}$. Find the largest possible cardinality of $\mathscr{A}$.