40-th Bulgarian Mathematical Olympiad 1991

Fourth Round

First Day – Sofia

1. Let M be a point on the altitude CD of an acute-angled triangle ABC, and K and L the orthogonal projections of M on AC and BC. Suppose that the incenter and circumcenter of the triangle lie on the segment KL.

   (a) Prove that $CD = R + r$, where R and r are the circumradius and inradius.

   (b) Find the minimum value of the ratio $CM : CD$.

2. Let K be a cube with edge n, where $n > 2$ is an even integer. Cube K is divided into n^3 unit cubes. We call any set of n^2 unit cubes lying on the same horizontal or vertical level a *layer*. We dispose of $n^3/4$ colors, in each of which we paint exactly 4 unit cubes. Prove that we can always select n unit cubes of distinct colors, no two of which lie on the same layer.

3. Prove that for every prime number $p \geq 5$,

   (a) p^3 divides $\binom{2p}{p} - 2$;

   (b) p^3 divides $\binom{kp}{p} - k$ for every natural number k.

Second Day – Sofia

4. Let $f(x)$ be a polynomial of degree n with real coefficients, having n (not necessarily distinct) real roots. Prove that for all real x,

$$f(x)f''(x) \leq f'(x)^2.$$

5. On a unit circle with center O, AB is an arc with the central angle $\alpha < 90^\circ$. Point H is the foot of the perpendicular from A to OB, T is a point on arc AB, and l is the tangent to the circle at T. The line l and the angle AHB form a triangle Δ.

   (a) Prove that the area of Δ is minimal when T is the midpoint of arc AB.

   (b) Prove that if S_α is the minimal area of Δ then the function S_α/α has a limit when $\alpha \to 0$ and find this limit.

6. White and black checkers are put on the squares of an $n \times n$ chessboard ($n \geq 2$) according to the following rule. Initially, a black checker is put on an arbitrary square. In every consequent step, a white checker is put on a free square, whereby all checkers on the squares neighboring by side are replaced by checkers of the opposite colors. This process is continued until there is a checker on every square. Prove that in the final configuration there is at least one black checker.

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