35-th Bulgarian Mathematical Olympiad 1986

Fourth Round

First Day

1. Find the smallest natural number n for which the number $n^2 - n + 11$ has exactly four prime factors (not necessarily distinct).

2. Let $f(x)$ be a quadric polynomial with two real roots in the interval $[-1, 1]$. Prove that if the maximum value of $|f(x)|$ in the interval $[-1, 1]$ is equal to 1, then the maximum value of $|f'(x)|$ in the interval $[-1, 1]$ is not less than 1.

3. A regular tetrahedron of unit edge is given. Find the volume of the maximal cube contained in the tetrahedron, whose one vertex lies in the feet of an altitude of the tetrahedron.

Second Day

4. Find the smallest integer $n \geq 3$ for which there exist an n-gon and a point within it such that, if a light bulb is placed at that point, on each side of the polygon there will be a point that is not lightened. Show that for this smallest value of n there always exist two points within the n-gon such that the bulbs placed at these points will lighten up the whole perimeter of the n-gon.

5. Let A be a fixed point on a circle k. Let B be any point on k and M be a point such that $AM : AB = m$ and $\angle BAM = \alpha$, where m and α are given. Find the locus of point M when B describes the circle k.

6. Let $0 < k < 1$ be a given real number and let $(a_n)_{n \geq 1}$ be an infinite sequence of real numbers which satisfies $a_{n+1} \leq \left(1 + \frac{k}{n}\right) a_n - 1$. Prove that there is an index t such that $a_t < 0$.

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