Bulgarian Mathematical Olympiad 1977, IV Round

First Day

1. Let for natural number n and real numbers α and x are satisfied inequalities $\alpha^{n+1} \leq x \leq 1$ and $0 < \alpha < 1$. Prove that

$$\prod_{k=1}^{n} \left| \frac{x-\alpha^k}{x+\alpha^k} \right| \leq \prod_{k=1}^{n} \left| \frac{1-\alpha^k}{1+\alpha^k} \right|$$

(7 points, Borislav Boyanov)

2. In the space are given n points and no four of them belongs to a common plane. Some of the points are connected with segments. It is known that four of the given points are vertices of tetrahedron which edges belong to the segments given. It is also known that common number of the segments, passing through vertices of tetrahedron is $2n$. Prove that there exists at least two tetrahedrons every one of which have a common wall with the first (initial) tetrahedron. (6 points, N. Nenov, N. Hadzhiivanov)

3. It is given truncated pyramid which bases are triangles with the base a triangle. The areas of the bases are B_1 and B_2 and the area of rounded surface is S. Prove that if there exists a plane parallel to the bases which intersection divides the pyramid to two truncated pyramids in whichmay be inscribed spheres then

$$S = \left(\sqrt{B_1} + \sqrt{B_2}\right)\left(\sqrt[4]{B_1} + \sqrt[4]{B_2}\right)^2$$

(7 points, G. Gantchev)

Second day

4. Vertices A and C of the quadrilateral $ABCD$ are fixed points of the circle k and each of the vertices B and D is moving to one of the arcs of k with ends A and C in such a way that $BC = CD$. Let M is the intersection point of AC and BD and F is the centre of circumscribed circle around $\triangle ABM$. Prove that the locus of F is an arc of a circle.

(7 points, J. Tabov)

5. Let $Q(x)$ is a non-zero polynomial and k is a natural number. Prove that the polynomial $P(x) = (x-1)^k Q(x)$ have at least $k+1$ non-zero coefficients. (7 points)

6. Pytagor's triangle is every right-angled triangle for which the lengths of two cathets and the length of the hypotenuse are integer numbers. We are observing all Pytagor's triangles in which may be inscribed a quadrangle with sidelength integer number, two of which sides lies on the cathets and one of the vertices of which lies on the hypotenuse of the triangle given. Find the sidelengths of the triangle with minimal surface from the observed triangles. (6 points, St. Dodunekov)

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