Bulgarian Mathematical Olympiad 1976, IV Round

First Day

1. In a circle with radius 1 is inscribed hexagon (convex). Prove that if the multiple of all diagonals that connects vertices of neighbour sides is equal to 27 then all angles of hexagon are equals.

   (5 points, V. Petkov, I. Tonov)

2. Find all polynomials $p(x)$ satisfying the condition:

$$p(x^2-2x)=(p(x-2))^2$$

   (7 points)

3. In the space is given a tetrahedron with length of the edge 2. Prove that distances from a random point M to all of the vertices of the tetrahedron are integer numbers if and only if M is a vertex of tetrahedron.

   (8 points, J. Tabov)

Second day

4. Let $0<x_1\le x_2\le\cdots\le x_n$. Prove that

$$\frac{x_1}{x_2}+\frac{x_2}{x_3}+\cdots+\frac{x_{n-1}}{x_n}+\frac{x_n}{x_1}\ge\frac{x_2}{x_1}+\frac{x_3}{x_2}+\cdots+\frac{x_n}{x_{n-1}}+\frac{x_1}{x_n}$$

   (7 points, I. Tonov)

5. It is given a tetrahedron $ABCD$ and a plane α intersecting the theree edges passing through D. Prove that α divides surface of tetrahedron to two partsproportional to the volumes of the bodies formed if and only if α is passing through the center of the inscribed tetrahedron sphere.

   (6 points, H. Lesov)

6. It is given a plane with coordinate system with a beginning at the point O. $A(n)$, when n is a natural number is a count of the points with whole coordinates which distances to O is less than or equal to n.

   (a) Find

$$\ell=\lim_{n\to\infty}\frac{A(n)}{n^2}$$

   (b) For which β $(1<\beta<2)$ there exists the limit

$$\lim_{n\to\infty}\frac{A(n)-\pi n^2}{n^\beta}$$

   (7 points)

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