Bulgarian Mathematical Olympiad 1973, IV Round

First Day

1. Let the sequence $a_1, a_2, \dots, a_n, \dots$ is defined by the conditions: $a_1 = 2$ and $a_{n+1} = a_n^2 - a_n + 1$ $(n = 1, 2, \dots)$. Prove that:

   (a) a_m and a_n are relatively prime numbers when $m \neq n$.

   (b) $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \frac{1}{a_k} = 1$.

   (Iv. Tonov)

2. Let the numbers a_1, a_2, a_3, a_4 form an arithmetic progression with difference $d \neq 0$. Prove that there are no exists geometric progressions b_1, b_2, b_3, b_4 and c_1, c_2, c_3, c_4 such that:

$$a_1 = b_1 + c_1,\ a_2 = b_2 + c_2,\ a_3 = b_3 + c_3,\ a_4 = b_4 + c_4$$

3. Let $a_1, a_2, \dots, a_n$ are different integer numbers in the range: $[100, 200]$ for which: $a_1 + a_2 + \cdots + a_n \geq 11100$. Prove that it can be found at least number from the given in the representation of decimal system on which there are at least two equal (same) digits. (L. Davidov)

Second day

4. Find all functions $f(x)$ defined in the range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ they can be differentiated for $x = 0$ and satisfy the condition:

$$f(x) = \frac{1}{2}\left(1 + \frac{1}{\cos x}\right) f\left(\frac{x}{2}\right)$$

   for every x in the range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. (L. Davidov)

5. Let the line ℓ intersects the sides AC, BC of the triangle ABC respectively at the points E and F. Prove that the line ℓ is passing through the incenter of the triangle ABC if and only if the following equality is true:

$$BC \cdot \frac{AE}{CE} + AC \cdot \frac{BF}{CF} = AB$$

   (H. Lesov)

6. In the tetrahedron $ABCD$, E and F are the middles of BC and AD, G is the middle of the segment EF. Construct a plane through G intersecting the segments AB, AC, AD in the points M, N, P respectively in such a way that the sum of the volumes of the tetrahedrons $BMNP$, $CMNP$ and $DMNP$ to be a minimal. (Hr. Lesov)

www.imomath.com

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
LaTeX by Borislav Mirchev
www.imomath.com