Bulgarian Mathematical Olympiad 1972, IV Round

First Day

1. Prove that there are no exists integer numbers a, b, c such that for every integer number x the number: $A=(x+a)(x+b)(x+c)-x^3-1$ is divisible by 9. (Iv. Tonov)

2. Solve the system of equations:

$$\begin{cases} \sqrt{\frac{y(t-y)}{t-x}-\frac{4}{x}}+\sqrt{\frac{z(t-z)}{t-x}-\frac{4}{x}}=\sqrt{x} \\ \sqrt{\frac{z(t-z)}{t-y}-\frac{4}{y}}+\sqrt{\frac{x(t-x)}{t-y}-\frac{4}{y}}=\sqrt{y} \\ \sqrt{\frac{x(t-x)}{t-z}-\frac{4}{z}}+\sqrt{\frac{y(t-y)}{t-z}-\frac{4}{z}}=\sqrt{z} \\ x+y+z=2t \end{cases}$$

   if the following conditions are satisfied: $0<x<t$, $0<y<t$, $0<z<t$.
   (Hr. Lesov)

3. Prove the equality:

$$\sum_{k=1}^{n-1}\frac{1}{\sin^2\frac{(2k+1)\pi}{2n}}=n^2$$

   where n is a natural number. (Hr. Lesov)

Second day

4. Find maximal possible count of points which lying in or over a circle with radii R in such a way that the distance between every two points is greater than: $R\sqrt{2}$. (Hr. Lesov)

5. In a circle with radii R is inscribed a quadrilateral with perpendicular diagonals. From the intersecting point of the diagonals are drawn perpendiculars to the sides of the quadrilateral.

   (a) prove that the feets of these perpendiculars P_1, P_2, P_3, P_4 are vertices of the quadrilateral that is inscribed and circumscribed.

   (b) Prove the inequalities: $2r_1 \leq \sqrt{2}R_1 \leq R$ where R_1 and r_1 are radiuces respectively of the circumcircle and inscircle to the quadrilateral: $P_1P_2P_3P_4$. When does equalities holds?

   (Hr. Lesov)

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6. It is given a tetrahedron $ABCD$ for which two points of opposite edges are mutually perpendicular. Prove that:

   (a) the four altitudes of $ABCD$ intersects at a common point H;

   (b) $AH + BH + CH + DH < p + 2R$, where p is the sum of the lengths of all edges of $ABCD$ and R is the radii of circumscribed around $ABCD$ sphere.

(Hr. Lesov)

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