Bulgarian Mathematical Olympiad 1968, IV Round

First Day

1. Find all possible natural values of k for which the system
$$\begin{cases} x_1+x_2+\cdots+x_k=9 \\ \dfrac{1}{x_1}+\dfrac{1}{x_2}+\cdots+\dfrac{1}{x_k}=1 \end{cases}$$
have solutions in positive numbers. Find these solutions.
(6 points, I. Dimovski)

2. Find all functions $f(x)$, defined for every x, y satisfying the equality
$$xf(y)+yf(x)=(x+y)f(x)f(y)$$
for every x, y. Prove that exactly two of them are continuous.
(6 points, I. Dimovski)

3. Prove that a binomial coefficient $\binom{n}{k}$ is odd if and only if all digits 1 of k, when k is written in binary digit system are on the same positions when n is written in binary system. (8 points, I. Dimovski)

Second day

4. Over the line g are given the segment AB and a point C external for AB. Prove that over g there exists at least one pair of points P, Q symmetricalwith respect to C, which divide the segment AB internally and externally in the same ratios, i.e.
$$\frac{PA}{PB}=\frac{QA}{QB} \tag{1}$$
Opposite if A, B, P, Q are such points from the line g for which is satisfyied (1), prove that the middle point C of the segment PQ is external point for the segment AB. (6 points, K. Petrov)

5. The point M is internal for the tetrahedron $ABCD$ and the intersection points of the lines AM, BM, CM and DM with the opposite walls are denoted with A_1, B_1, C_1, D_1 respectively. It is given also that the ratios $\frac{MA}{MA_1}$, $\frac{MB}{MB_1}$, $\frac{MC}{MC_1}$ and $\frac{MD}{MD_1}$ are equal to the same number k. Find all possible values of k. (8 points, K. Petrov)

6. Find the kind of the triangle if
$$\frac{a\cos\alpha+b\cos\beta+c\cos\gamma}{a\sin\alpha+b\sin\beta+c\sin\gamma}=\frac{2p}{9R}$$
(α, β, γ are the measures of the angles, a, b, c, p, R are the lengths of the sides, the p-semiperimeter, the radii of the circumcircle of the triangle).
(6 points, K. Petrov)

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
LaTeX by Borislav Mirchev
www.imomath.com