Bulgarian Mathematical Olympiad 1967, IV Round

1. The numbers 12, 14, 37, 65 are one of the solutions of the equation:

$$xy - xz + yt = 182$$

What number of what letter corresponds? (5 points)

2. Prove that:

   (a) if $y < \frac{1}{2}$ and $n \geq 3$ is a natural number then: $(y+1)^n \geq y^n + (1+2y)^{\frac{n}{2}}$;

   (b) if x, y, z and $n \geq 3$ are natural numbers for which: $x^2 - 1 \leq 2y$ then $x^n + y^n \neq z^n$.

(9 points)

3. It is given a right-angled triangle ABC and its circumcircle k.

   (a) prove that the radii of the circle k_1 tangent to the cathets of the triangle and to the circle k is equal to the diameter of the incircle of the triangle ABC.

   (b) on the circle k may be found a point M for which the sum $MA + MB + MC$ is biggest possible.

(11 points)

4. Outside of the plane of the triangle ABC is given point D.

   (a) prove that if the segment DA is perpendicular to the plane ABC then orthogonal projection of the orthocenter of the triangle ABC on the plane BCD coincides with the orthocenter of the triangle BCD.

   (b) for all tetrahedrons $ABCD$ with base, the triangle ABC with smallest of the four heights that from the vertex D, find the locus of the foot of that height.

(10 points)

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
LaTeX by Borislav Mirchev
www.imomath.com