Bulgarian Mathematical Olympiad 1966, IV Round

1. Prove that the equality:
$$3x(x-3y)=y^2+z^2$$
doesn't have other integer solutions except $x=0$, $y=0$, $z=0$.
(5 points)

2. Prove that for every four positive numbers a, b, c, d is true the following inequality:
$$\sqrt{\frac{a^2+b^2+c^2+d^2}{4}} \geqslant \sqrt[3]{\frac{abc+abd+acd+bcd}{4}}$$
(7 points)

3. (a) In the plane of the triangle ABC find a point with the following property: its symmetrical points with respect to middle points of the sides of the triangle lies on the circumscribed circle.
   (b) Construct the triangle ABC if it is known the positions of the orthocenter H, middle point of the side AB and the middle point of the segment joining the foots of the heights through vertices A and B.

   (9 points)

4. It is given a tetrahedron with vertices A, B,
   (a) Prove that there exists vertex of tetrahedron with the following property: the three edges of that tetrahedron can be constructed a triangle.
   (b) Over the edges DA, DB and DC are given the points M, N and P for which:
   $$DM=\frac{DA}{n},\quad DN=\frac{DB}{n+1},\quad DP=\frac{DC}{n+2}$$
   where n is a natural number. The plane defined by the points M, N and P is α_n. Prove that all planes α_n, $(n=1,2,3,\ldots)$ passes through a single straight line.

   (9 points)

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