Bulgarian Mathematical Olympiad 1965, IV Round

1. The numbers 2, 3, 7 have the property that the multiple of any two of them increased by 1 is divisible of the third number. Prove that this triple of integer numbers greater than 1 is the only triple with the given property. (6 points)

2. Prove the inequality:

$$\left(1+\sin^2\alpha\right)^n+\left(1+\cos^2\alpha\right)^n \geqslant 2\left(\frac{3}{2}\right)^n$$

is true for every natural number n. When does equality holds? (5 points)

3. In the triangle ABC angle bisector CD intersects circumscribed around ABC circle at the point K.

   (a) Prove the equalities:

$$\frac{1}{JD}-\frac{1}{JK}=\frac{1}{CJ} \qquad , \qquad \frac{CJ}{JD}-\frac{JD}{DK}=1$$

   where J is the centre of the inscribed circle.

   (b) On the segment CK is chosen a random point P with projections on AC, BC, AB respectively: P_1, P_2, P_3. The lines PP_3 and P_1P_2 intersects at a point M. Find the locus of M when P is moving around the CK segment.

   (9 points)

4. In the space are given crossed lines s and t such that $\angle(s,t)=60^\circ$ and a segment AB perpendicular to them. On AB is chosen a point C for which $AC:CB=2:1$ and the points M and N are moving on the lines s and t in such a way that $AM=2BN$. Prove that[1]:

   (a) the segment MN is perpendicular to t;

   (b) the plane α, perpendicular to AB in point C intersects the plane CMN on fixed line ℓ with given direction in respect to s and t;

   (c) reverse, all planes passing by *ell* and perpendicular to AB intersects the lines s and t respectively at points M and N for which $AM=2BN$ and $MN\perp t$.

   (6 points)

[1] In the statement should be said that vectors $\overrightarrow{AM}$ and $\overrightarrow{BM}$ have the angle between them 60°

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