Bulgarian Mathematical Olympiad 1963, IV Round

1. Find all three-digit numbers which remainders after division by 11 give quotient, equal to the sum of it's digits squares. (4 points)

2. It is given the equation $x^2 + px + 1 = 0$, with roots x_1 and x_2;

   (a) find a second-degree equation with roots y_1, y_2 satisfying the conditions: $y_1 = x_1(1 - x_1)$, $y_2 = x_2(1 - x_2)$;

   (b) find all possible values of the real parameter p such that the roots of the new equation lies between -2 and 1.

   (5 points)

3. In the trapezium $ABCD$ with on the non-base segment AB is chosen a random point M. Through the points M, A, D and M, B, C are drawn circles k_1 and k_2 with centers O_1 and O_2. Prove that:

   (a) the second intersection point N of k_1 and k_2 lies on the other non-base segment CD or on its continuation;

   (b) the length of the line O_1O_2 doesn't depend of the situation on M over AB;

   (c) the triangles O_1MO_2 and DMC are similar. Find such a position of M over AB that makes k_1 and k_2 with the same radii.

   (6 points)

4. In the tetrahedron $ABCD$ three of the sides are right-angled triangles and the second in not an obtuse triangle. Prove that:

   (a) the fourth wall of the tetrahedron is right-angled triangle if and only if exactly two of the plane angles having common vertex with the some of vertices of the tetrahedron are equal.

   (b) when all four walls of the tetrahedron are right-angled triangles its volume is equal to $\frac{1}{6}$ multiplied by the multiple of three shortest edges not lying on the same wall.

   (5 points)

   REMARK FOR (B) - more correct statement should be: *cdots its volume is equal to $\frac{1}{6}$ multiplied by the multiple of two shortest edges and an edge not lying on the same wall.*

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
LaTeX by Borislav Mirchev
www.imomath.com