57-th Bulgarian Mathematical Olympiad 2008, National Round

First Day, 17-th May 2008

1. Let ABC is acute-angled triangle and CL is its internal angle bisector and $L \in AB$. The point P belongs to the segment CL in such a way that $\angle APB = \pi - \frac{1}{2}\angle ACB$. Let k_1 and k_2 are the circumcircles of $\triangle APC$ and $\triangle BPC$. $BP \cap k_1 = Q$ and $BP \cap k_2 = R$. The tangents to k_1 in Q and to k_2 in B intersects at the point S and the tangents to k_1 at R and to k_2 at A intersects at the point T. Prove that $AS = BT$.

2. Are there exists 2008 non-intersecting arithmetic progressions composed from natural numbers such that each of them contains a prime number greater than 2008 and the numbers that doesn't belongs to (some of) the progressions are finite number?

3. Let $n \in \mathbb{N}$ and $0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n \leq \pi$ and $b_1, b_2, \ldots, b_n$ are real numbers for which the following inequality is satisfied:

$$\left| \sum_{i=1}^{n} b_i \cos(k\alpha_i) \right| < \frac{1}{k}$$

for all $k \in \mathbb{N}$. Prove that $b_1 = b_2 = \cdots = b_n = 0$.

Second day, 18-th May 2008

4. Find the smallest natural number k for which there exists natural numbers m and n such that $1324 + 279m + 5^n$ is k-th power of some natural number.

5. Let n is a fixed natural number. Find all natural numbers m for which

$$\frac{1}{a^n} + \frac{1}{b^n} \geq a^m + b^m$$

is satisfied for every two positive numbers a and b with sum equal to 2.

6. Let M is the set of the integer numbers from the range $[-n, n]$. The subset P of M is called *base subset* if every number from M can be expressed as a sum of some different numbers from P. Find the smallest natural number k such that every k numbers that belongs to M form a *base subset*.

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