48-th Bulgarian Mathematical Olympiad 1999

Third Round – April 17–18, 1999

First Day

1. Find all triples (x,y,z) of natural numbers such that y is prime, y and 3 do not divide z, and $x^3 - y^3 = z^2$. *(N. Nikolov)*

2. A convex quadrilateral of area S is inscribed in a circle whose center is inside the quadrilateral. Prove that the area of the quadrilateral whose vertices are at the projections of the intersection point of the diagonals onto the sides does not exceed $S/2$. *(H. Lesov)*

3. In a competition 8 judges marked the contestants by *yes* or *no*. It is known that for any two contestants, two contestants gave both a *yes*; two judges gave the first one a *yes* and the second one a *no*; two judges gave the first one a *no* and the second one a *yes*; and two judges gave both a *no*. What is the greatest possible number of contestants? *(E. Kolev)*

Second Day

4. Find all pairs of integers (x,y) such that $x^3 = y^3 + 2y^2 + 1$. *(N. Nikolov, E. Kolev)*

5. In a triangle ABC, B_1 and C_1 are points on sides AC and AB respectively. The lines BB_1 and CC_1 meet at D. Prove that the quadrilateral $ABCD$ is circumscribable around a circle if and only if the incircles of triangles ABD and ACD are tangent. *(R. Kozarev, N. Nikolov)*

6. An equilateral triangle of side 1 is covered by six circles of the same radius r. Prove that $r \geq \sqrt{3}/10$. *(N. Nikolov, E. Kolev)*