47-th Bulgarian Mathematical Olympiad 1998

Third Round – April 25–26, 1998

First Day

1. Find the least integer $n \geq 3$ with the following property: For any coloring of n different points $A_1, A_2, \dots, A_n$ on a line such that $A_1A_2 = A_2A_3 = \cdots = A_{n-1}A_n$ in two colors, there are three points A_i, A_j, A_{2j-i} which have the same color.

2. Let $ABCD$ be a quadrilateral such that $AD = CD$ and $\angle DAB = \angle ABC < 90^\circ$. The line passing through D and the midpoint of BC intersects AB at point E. Prove that $\angle BEC = \angle DAC$.

3. Prove that there is no function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that
$$f(x)^2 \geq f(x+y)(f(x)+y) \quad \text{for all } x, y > 0.$$

Second Day

4. Let $f(x) = x^3 - 3x + 1$. Find the number of different real solutions of the equation $f(f(x)) = 0$.

5. A convex pentagon $ABCDE$ is inscribed in a circle with radius R. Let r_{XYZ} denote the inradius of a triangle XYZ. Prove that
   (a) $\cos\angle CAB + \cos\angle ABC + \cos\angle BCA = 1 + \frac{r_{ABC}}{R}$;
   (b) if $r_{ABC} = r_{AED}$ and $r_{ABD} = r_{AEC}$, then $\triangle ABC \cong \triangle AED$.

6. Show that the equation $x^2y^2 = z^2(z^2 - x^2 - y^2)$ has no solution in positive integers.

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