46-th Bulgarian Mathematical Olympiad 1997

Third Round – April 1997

First Day

1. Find all natural numbers a,b,c such that the roots of the equations

$$x^2-2ax+b=0,\quad x^2-2bx+c=0,\quad x^2-cx+a=0$$

are natural numbers.

2. In a cyclic quadrilateral $ABCD$, lines AD and BC meet at E, and diagonals AC and BD meet at F. If M and N are the midpoints of AB and CD, prove that

$$\frac{MN}{EF}=\frac{1}{2}\left|\frac{AB}{CD}-\frac{CD}{AB}\right|.$$

3. Prove that the equation

$$x^2+y^2+z^2+3(x+y+z)+5=0$$

has no solution in rational numbers.

Second Day

4. Find all continuous functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x)=f\left(x^2+\frac{1}{4}\right)\quad \text{for all real } x.$$

5. Two unit squares $\mathscr{K}_1$ and $\mathscr{K}_2$ with centers M and N respectively are placed in the plane so that $MN=4$, two sides of $\mathscr{K}_1$ are parallel to MN, and one diagonal of $\mathscr{K}_2$ lies on MN. Find the locus of midpoints of segments XY, where X is an interior point of $\mathscr{K}_1$ and Y is an interior point of $\mathscr{K}_2$.

6. Find the number of nonempty subsets of $S_n=\{1,2,\dots,n\}$ which contain no two consecutive numbers.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com