45-th Bulgarian Mathematical Olympiad 1996

Third Round

First Day

1. Prove that for all positive integers $n \geq 3$ there exist odd positive integers x_n, y_n such that $7x_n^2 + y_n^2 = 2^n$.

2. The circles k_1 and k_2 with centers O_1 and O_2 respectively are externally tangent at point C, and the circle k with center O is externally tangent to k_1 and k_2. Let l be the common tangent of k_1 and k_2 at C, and let AB be the diameter of k which is perpendicular to l, where A and O_1 lie on the same side of l. Prove that the lines AO_2, BO_1 and l have a common point.

3. (a) Find the maximum value of $y = |4x^3 - 3x|$ for $-1 \leq x \leq 1$.

   (b) Let a, b, c be real numbers and M be the maximum value of $y = |4x^3 + ax^2 + bx + c|$ for $-1 \leq x \leq 1$. Show that $M \geq 1$. For which a, b, c does the equality hold?

Second Day

4. Suppose that the real numbers $a_1, a_2, \dots, a_n$ form an arithmetic progression, and that some permutation $a_{i_1}, \dots, a_{i_n}$ of these numbers forms a geometric progression. Find the numbers $a_1, \dots, a_n$ if they are different and the biggest among them is equal to 1996.

5. A convex quadrilateral $ABCD$ with $\angle ABC + \angle BCD < 180^\circ$ is given. The lines AB and CD meet at E. Prove that $\angle ABC = \angle ADC$ if and only if

$$AC^2 = CD \cdot CE - AB \cdot AE.$$

6. An $m \times n$ rectangle $(m, n > 1)$ is divided into mn unit squares. In how many ways ways can two of the squares be cut off so that the remaning part of the rectangle can be covered with dominoes 2×1.

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