44-th Bulgarian Mathematical Olympiad 1995

Third Round

First Day

1. Let p and q be positive numbers such that the parabola $y = x^2 - 2px + q$ has no common point with the x-axis. Prove that there exist points A and B on the parabola such that AB is parallel to the x-axis and $\angle AOB = 90^\circ$, where O is the origin $(0,0)$, if and only if $p^2 < q \le 1/4$. Find the values p and q for which the segment AB is unique.

2. Let $A_1A_2 \dots A_7$, $B_1B_2 \dots B_7$, $C_1C_2 \dots C_7$ be regular heptagons with areas S_A, S_B, S_C, respectively, such that $A_1A_2 = B_1B_3 = C_1C_4$. Prove that
$$\frac{1}{2} < \frac{S_B + S_C}{S_A} < 2 - \sqrt{2}.$$

3. Let $n > 1$ be an integer. Find the number of permutations $(a_1, a_2, \dots, a_n)$ of the numbers $1, 2, \dots, n$ with the property that $a_i > a_{i+1}$ holds for only one index $i \in \{1, 2, \dots, n-1\}$.

Second Day

4. Let $n \ge 2$ and $0 \le x_i \le 1$ for $i = 1, 2, \dots, n$. Prove that
$$(x_1 + x_2 + \cdots + x_n) - (x_1x_2 + x_2x_3 + \cdots + x_nx_1) \le \left[\frac{n}{2}\right].$$
When does equality hold?

5. Let M be an interior point of the triangle ABC. The lines AM, BM, CM meet the opposite sides of the triangle at A_1, B_1, C_1 respectively. Prove that if M is the centroid of $\triangle A_1B_1C_1$, then M is the centroid of $\triangle ABC$.

6. Find all pairs of positive integers (x, y) for which $\dfrac{x^2 + y^2}{x - y}$ is an integer that divides 1995.

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