39-th Bulgarian Mathematical Olympiad 1990

Third Round

First Day

1. A quadrilateral $ABCD$ is inscribed in a circle. Points P and Q are chosen on the rays AB and AD respectively so that $AP = CD$ and $AQ = BC$. If the line AC intersects PQ at M, prove that $PM = MQ$.

2. Let M be the set of functions $f(x) = x^2 + ax + b$, where $a, b \in \mathbb{R}$. Prove that for every $f \in M$ the maximum value of $|f(x)|$ on the interval $[-1, 1]$ is not smaller than $1/2$. For which f is this maximum value equal to $1/2$?

3. Natural numbers $a_0, a_1, \ldots, a_8$ satisfy the condition $a_{n+1} = a_n^2 - a_n + 5$ for $n = 0, 1, \ldots, 7$. Prove that at least two of these numbers are not coprime.

Second Day

4. For each real parameter a, find the maximum and minimum values, if they exist, of the function
$$f(x) = \frac{x^2}{x^2 + ax + 1}.$$

5. Given three disjoint spheres inside each other, find the locus of the centers of the spheres which cut each of these three spheres in large circles.

6. Squares of an $m \times n$ chessboard are colored black and white in such a way that a king, starting from any square of the leftmost column, cannot reach any square of the rightmost column. Show that a rook can be placed at some square of the lowermost row, so that it can reach the uppermost row (not violating the chess rules).

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