38-th Bulgarian Mathematical Olympiad 1989

Third Round

First Day

1. Suppose that p and q are prime numbers such that

$$\sqrt{p^2+7pq+q^2}+\sqrt{p^2+14pq+q^2}$$

is an integer. Prove that $p=q$.

2. Prove that for every integer $n \geq 1$ the equation $x\cos x = 1$ has a unique solution in the interval $[2n\pi,(2n+1)\pi]$. Find $\lim_{n\to\infty}(x_{n+1}-x_n)$.

3. A line parallel to the side AB of a triangle ABC meets the sides AC and BC at M and P, respectively. The lines AP and BM intersect at D. Prove that the line passing through the orthocenters of the triangles ADM and BDP is perpendicular to CD.

Second Day

4. A convex n-gon ($n>3$) has the property that there exist $n-2$ diagonals of the n-gon, each of which bisects its area. Find all such n-gons.

5. A plane parallel to the base ABC of a tetrahedron $SABC$ cuts the edges SA,SB,SC at A_1,B_1,C_1, respectively. Let A_2,B_2,C_2 respectively be the midpoints of B_1C_1,C_1A_1,A_1B_1. Prove that if $AA_2 \perp B_1C_1$ and $BB_2 \perp C_1A_1$, then

   (a) $CC_2 \perp A_1B_1$;

   (b) The line through the orthocenter of $\triangle ABC$ and the circumcenter of $\triangle A_1B_1C_1$ is perpendicular to the plane ABC.

6. Let $0 \leq a_1 \leq a_2 \leq \cdots \leq a_5$ be real numbers. Denote

$$S=a_1+a_2+a_3+a_4+a_5, \quad P=\prod_{1\leq i\leq j\leq 5}(S-2(a_i+a_j)).$$

Prove that if $a_1+3a_2+a_4 \geq 3a_3+a_5$, then $P \leq (a_1a_2a_3a_4a_5)^2$.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com