37-th Bulgarian Mathematical Olympiad 1988

Third Round

First Day

1. A traveller came to an island on which every inhabitant is either a liar (always lies) or an honest person (always tells the truth). He talked to four islanders A,B,V,G, and A said ”exactly one of us four is a liar”, while B said ”all of us are liars”. Then the traveller asked V if A is a liar, but got an answer (”yes” or ”no”) from which he cannot conclude the truth about A. Is G a liar?

2. Given a triangle ABC and a segment m, construct a circle k passing through A and B such that the common chord of k and the incircle of $\triangle ABC$ is of the length m.

3. Let $u_1,u_2,\dots$ be an infinite decreasing geometric progression such that $u_3 = 8$ and $u_3+u_4+u_5 = 14$.

   (a) Find the ratio of this progression;

   (b) Prove that for each $c \in (0,1)$ the sequence (w_n) defined by $w_1 = u_1+c$ and $w_n = w_{n-1}(u_n+c)$ for $n > 1$ converges.

Second Day

4. Find all real values of the parameters p and q for which the polynomial

$$f(x) = x^4-\frac{8p^2}{q}x^3+4qx^2-3px+p^2$$

   has four positive roots. For such p and q find these roots.

5. (a) For every integer $n \geq 0$, $f(n)$ denotes the number of solutions of the equation $x+2y = n$ in nonnegative integers. Find the formula for $f(n)$.

   (b) For every integer $n \geq 0$, $g(n)$ denotes the number of solutions of the equation $x+2y+3z = n$ in nonnegative integers. Prove that $g(n) = g(n-3)+[n/2]+1$.

6. Let A_1,B_1,C_1 be the feet of the perpendiculars from the vertices A,B,C of a tetrahedron $ABCD$ to the opposite faces respectively. Prove that if A_1 and B_1 are the centroids of triangles BCD and CDA, and C_1 is the circumcenter of triangle DAB, then the tetrahedron $ABCD$ is regular.

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