36-th Bulgarian Mathematical Olympiad 1987

Third Round

First Day

1. Let a, b be real numbers with $a \geq -0.5$, $b \neq 0$ and $a/b > 1$. Prove that
$$\frac{2a^3+1}{4b(a-b)} \geq 3.$$

2. Prove that if two medians of an acute triangle are equal to two of its altitudes, then the triangle is equilateral.

3. Some diagonals of a regular polygon are drawn so that no two of them have a common point other than an endpoint. The vertices of the polygon which are endpoints of exactly one of the drawn diagonals are painted blue, while the others are painted red. Let n be the number of the polygons with vertices at the vertices of the given polygon which have at least one red vertex, and let m be the number of polygons whose all vertices are blue. Show that $n > 3m$.

Second Day

4. Determine all pairs (x, y) of natural numbers such that
$$x(x+1)(x+7)(x+8) = y^2.$$

5. Prove that for every real parameter $k < 6$ the system
$$x^2 + y^2 = k^2, \quad y = |x^2 - kx| - k$$
has an odd number of real solutions.

6. Let V_n be the minimum possible volume of a regular n-gonal pyramid in which the distance from the center of the base to a lateral face is 1.
   (a) Express V_n in terms of n.
   (b) Find $\lim_{n\to\infty} V_n$.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com