34-th Bulgarian Mathematical Olympiad 1985

Third Round – April 13-14

First Day

1. If n and k are natural numbers, prove that n^5+1 divides
$$(n^4-1)^k(n^3-n^2+n-1)+(n+1)n^{4k-1}.$$

2. Find for which values of the real parameter a the equation $\lg 2x \lg 3x = a$ has two distinct positive solutions, and determine the product of these two solutions.

3. In a tetrahedron $ABCD$, the midpoints of the edges AB and CD and the incenter lie on a line. Prove that the circumcenter of the tetrahedron also lies on this line.

Second Day

4. Let a_n, b_n be natural numbers such that $a_n + b_n\sqrt{2} = (2+\sqrt{2})^n$, where $n \in \mathbb{N}$. Prove that the limit $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists, and find it.

5. A triangle ABC of area S is inscribed in a circle k of radius 1. The orthogonal projections of the incenter I of $\triangle ABC$ on BC, CA, AB are A_1, B_1, C_1 respectively, and S_1 is the area of $\triangle A_1B_1C_1$. If AI meets k again at A_2, prove that $4S_1 = AI \cdot A_2B \cdot S$.

6. In the plane are given five points with the property that among any four of them, some three are vertices of an equilateral triangle.
   (a) Prove that some four of these points are vertices of a rhombus with the acute angle equal to 60°.
   (b) Find the number of equilateral triangles with the vertices in these five points.

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