33-rd Bulgarian Mathematical Olympiad 1984

Third Round

First Day

1. For each natural number n, let a_n be the number of perfect squares among the numbers $2, 9, 16, \dots, 7n+2$, and let b_n be the number of perfect squares among $1, 4, 7, \dots, 3n+1$.

   (a) Find a_{1984};

   (b) Find the smallest n such that $b_n = a_{1984}$.

2. In an isosceles trapezoid $ABCD$ with bases AB and CD, M and N are the feet of the perpendiculars from D to AB and AC respectively, and P is the midpoint of CD. Prove that if the points M, N, P are collinear, then $\angle ACB = 90^\circ$.

3. Let a, b, p, q, α, and $c < 1$ be positive numbers.

   (a) Find the minimum of $f(x) = \dfrac{x^{\alpha+1}}{c^\alpha} + \dfrac{(1-x)^{\alpha+1}}{(1-c)^\alpha}$ for $x \in (0,1)$.

   (b) Prove the inequality $\dfrac{a^{\alpha+1}}{p^\alpha} + \dfrac{b^{\alpha+1}}{q^\alpha} \geq \dfrac{(a+b)^{\alpha+1}}{(p+q)^\alpha}$.

Second Day

4. Solve the equation
$$\log_{3x+4}(4x^2+4x+1) + \log_{2x+1}(6x^2+11x+4) = 4.$$

5. Find all natural numbers a, b, c and m such that there is a right triangle with sides a, b, c and with the perimeter equal to m times the area.

6. Suppose that the feet of the altitudes from C and D of a tetrahedron $ABCD$ are the incenters of the opposite faces, and that $AB = BD$. Prove that the tetrahedron $ABCD$ is regular.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com