31-st Bulgarian Mathematical Olympiad 1982

Third Round

First Day

1. Find all representations of number 1982 as the sum of several natural numbers $a_1, a_2, \dots, a_n$ $(n > 1)$ which form an arithmetic progression with odd common difference, such that a_i is divisible by $i+1$ for all i.

2. In a non-isosceles triangle $A_1A_2A_3$, let B_{ij} denote the reflection of A_i in the bisector of the angle at A_j. Prove that the three lines $A_{12}A_{21}$, $A_{13}A_{31}$ and $A_{23}A_{32}$ are parallel.

3. Find the set of numbers x in the interval $[0, 2\pi]$ which satisfy

$$\sqrt{\frac{4}{3} - \sqrt{\frac{4}{3}}}\sin x \cos x + \left(1 + \sqrt{\frac{1}{3}}\right)\sin^2 x \geq 1.$$

Second Day

4. Given a natural number n and a real number $a > 1/2$, consider the function

$$f(x) = \frac{1}{(1-x)^n} - \frac{1}{(1+x)^n} - 2nx.$$

   (a) Prove that for $0 < x < 1$, $f''(x) > 0$, $f'(x) > 0$ and $f(x) > 0$.

   (b) Prove that $\dfrac{n}{a^{n+1}} < \dfrac{1}{(a-\frac{1}{2})^n} - \dfrac{1}{(a+\frac{1}{2})^n}$.

5. On a chess tournament, two participants withdrew after the fifth round. Given that the total number of matches played on the tournament was 38, decide if these two players played against each other.

6. A sphere $\mathscr{S}$ with radius r passes through the center of a sphere Σ with radius $R > 2r$. If a chord EF of Σ is tangent to $\mathscr{S}$ at T, show that

$$TE^2 + TF^2 \leq 2R^2 + r^2.$$

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