Bulgarian Mathematical Olympiad 1977, III Round

First Day

1. It is given a pyramid with a base quadrilateral. In each of the walls of the pyramid is inscribed a circle. It is also known that inscribed circles of each two of the walls with common edge have a common point on that edge. Prove that four points in that inscribed in the four walls circles are tangent to the base lies on a common circle. (Here walls are the four side walls of the pyramid and the base of the pyramid is not a wall).

   (7 points)

2. Find all integer numbers x for which x^2+1 divides x^3-8x^2+2x without remainder. (6 points)

3. Prove that when the number $\left(7+4\sqrt{3}\right)^n$, $n \geq 1$, $n \in \mathbb{N}$ is written in decimal system the digit 9 occurs in it at least n times after the decimal point. (7 points)

Second day

4. Find all real solutions of the system:

$$\begin{cases} \dfrac{2x^2}{1+x^2} = y \\ \dfrac{2y^2}{1+y^2} = z \\ \dfrac{2z^2}{1+z^2} = x \end{cases}$$

   (7 points)

5. There are given two circumferences k_1 and k_2 with centers O_1 and O_2 respectively with different radiis that are tangent outside each other at a point A. It is given a point M inside k_1, not lying on the line O_1O_2. Construct a line ℓ that passes through M and for which circumscribed circle with vertices A and two of the common points of ℓ with k_1 and ℓ with k_2 is tangent to the line O_1O_2. (7 points, Jordan Tabov)

6. In a group of people two mans X, Y are named *non-directly known* if they not know each other (themself) personally or if exists a chain of people $Z_1, Z_2, \ldots, Z_p$ such that X and Z_1 are known, Z_1 and Z_2 are known, ..., Z_p and Y are known. Let the group consists of 134 persons and for each 8 of them at least two are *non-directly known*. Prove that there exists a group of 20 people every two of which are *non-directly known*.

   (6 points, N. Nenov, N. Hadzhiivanov)

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