Bulgarian Mathematical Olympiad 1975, III Round

First Day

1. Let n is an odd natural number and $a_1, a_2, \ldots, a_n$ is a permutation of the numbers $1, 2, \ldots, n$. Prove that the number

$$(a_1 - 1)(a_2 - 3)\cdots(a_n - n)$$

is an even number. (L. Davidov)

2. Let m, n, p are three sides of a billiard table with the shape of an equilateral triangle. A ball is situated at the middle of m side and is directed to the side n under angle α (the angle of the trajectory of the ball and n is α). For which values of α the ball after its reflection on n will reach the side p and after its reflection will reach the side m? (V. Petkof)

3. Prove that the number $2^{147} - 1$ is divisible by 343. (V. Chukanov)

Second day

4. Find all polynomials $f(x)$, satisfying the conditions $f(2x) = f'(x)f''(x)$. (L. Davidov)

5. Calculate:

$$\left[\frac{1}{2}\left(1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{5}} + \cdots + \frac{1}{\sqrt{10000}}\right)\right]$$

where $[\alpha]$ is the integer part of the number α (biggest integer number not bigger than α). A possible way to calculate that number is to prove and use the inequality

$$2\left(\sqrt{n+1} - \sqrt{n}\right) < \frac{1}{\sqrt{n}} < 2\left(\sqrt{n} - \sqrt{n-1}\right)$$

for all integer numbers n. (I. Prodanov)

6. In a regular n-angled truncated pyramid we may inscribe a sphere touching all walls and can be found other sphere touching all edges of pyramid

   (a) Prove that $n = 3$;

   (b) find dihedral angle between a surrounding wall and the biggerbase.

(V. Petkov)

www.imomath.com

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
LaTeX by Borislav Mirchev
www.imomath.com