Bulgarian Mathematical Olympiad 1974, III Round

First Day

1. The sequence $\{a_n\}$ is defined in the following way: $a_0 = 2$ and

$$a_{n+1} = a_n + \frac{1}{a_n} \quad , \quad n = 0, 1, 2, \dots$$

Prove that $62 < a_{1974} < 77$. (5 Points, I. Tonov)

2. It's given the triangle ABC. In its sides externally are constructed similar triangles ABK, BCL, CAM (it is know that $AB : BC : CA = KA : LB : MC$). Prove that the centers of gravity of the triangles ABC and KLM coincides. (7 Points, L. Davidov)

3. Let n and k are natural numbers such that $k \geq 2$. Prove that there exists n sequential natural numbers, such that every one of them may be presented as a multiple of at least k prime multipliers. (8 Points, V. Chukanov)

Second day

4. Find the natural number x defined by the equality:

$$\left[\sqrt[3]{1}\right] + \left[\sqrt[3]{2}\right] + \cdots + \left[\sqrt[3]{x^3 - 1}\right] = 400$$

(6 Points, V. Petnov)

5. In a cube with edge 9 are *thrown* 40 regular phrisms with side of the base 1,5 and a height not greater than 1,4. Prove that there exists a sphere with radius 0,5 lying in the cube and not having common points with the prisms. (6 Points, H. Lesov)

6. In a plane are given circle k with centre O and point P lying outside k. There are constructed tangents PQ and PR from P to k and on the smaller arc QR is chosen a random point (B, $B \neq Q$, $B \neq R$). Through point B is constructed a tangent to k intersecting PQ and PR respectively at points A and C. Prove that the length of the segment QR is equal to the minimal perimeter of inscribed AOC triangles. (8 points, L. Davidov)

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