Bulgarian Mathematical Olympiad 1970, III Round

First Day

1. Prove the the inequality
$$\frac{1-a}{1+a}+\frac{1-b}{1+b}+\frac{1-c}{1+c}\geq\frac{3}{2}$$
where $a\geq 0$, $b\geq 0$, $c\geq 0$ and $a+b+c=1$. (5 Points)

2. There are given the numbers $a=123456789$ and $b=987654321$. Find:
   (a) biggest common divisor of a and b;
   (b) remainder after division of the smallest common multiple of a and b to 11.

   (8 Points)

3. Points of plane are divided to three groups *white*, *greeen*, *red*. Prove that there exists at least one pair of points with the same color (from the same group), which have a distance to each other equal to 1. (7 Points)

Second day

4. In the triangle ABC is given a point M and through M are drawn lines, parallel to the sides of the triangle. These lines cut from the triangle three smaller triangles in such a way that one of the vertices of each triangle is a vertex of the biggest triangle ABC. Let P_a, P_b, P_c are perimeters of the given triangle and S_a, S_b, S_c are the areas of these triangles. P and S are the perimeter and the area of thye triangle ABC. Prove that:
   (a) $P=\dfrac{P_a+P_b+P_c}{2}$;
   (b) $\sqrt{S}=\dfrac{\sqrt{S_a}+\sqrt{S_b}+\sqrt{S_c}}{2}$.

   (5 Points)

5. Calculate without using logaritmic table or other additional tools
$$S_n(\alpha)=\frac{\cos 2\alpha}{\sin 3\alpha}+\frac{\cos 6\alpha}{\sin 9\alpha}+\cdots+\frac{\cos 2\cdot 3^{n-1}\alpha}{\sin 3^n\alpha}$$
for $\alpha=18^\circ$, where n is a natural number in the form $1+4k$. (7 Points)

6. It is given quadrilateral prism $ABCDA_1B_1C_1D_1$, for which the smallest distance between AA_1 and BD_1 is 8m and the distance from the vertex A_1 to the plane of the triangle ACB_1 is $\frac{24}{\sqrt{13}}$m. Through middlepoints of the edges AB and BC is constructed intersection which divides the axis of the prism in ratio $1:3$ from bottom base ($ABCD$):

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(a) what is the shape of the intersection;

(b) calculate the area of the intersection.

(8 Points)

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