Bulgarian Mathematical Olympiad 1969, III Round

First Day

1. Prove that for every natural number n the number $N = 1 + 2^{2\cdot 5^n}$ is divisible by 5^{n+1}. (6 Points)

2. Prove that the polynomial $f(x) = x^5 - x + a$, where a is an integer number which is not divisible by 5, cannot be written as a product of two polynomials with lower degree. (8 Points)

3. There are given 20 different natural numbers smaller than 70. Prove that among their differences there are two equals. (6 Points)

Second day

4. It is given acute-angled triangle with sides a, b, c. Let p, r and R are semiperimeter, radii of inscribed and radii of circumscribed circles respectively. It's center of gravity is also a midpoint of the segment with edges incenter and circumcenter. Prove that the following equality is true:

$$7\left(a^2+b^2+c^2\right) = 12p^2 + 9R(R-6r)$$

(7 Points)

5. In the triangle pyramid $OABC$ with base ABC, the edges OA, OB, OC are mutually perpendicular (each two of them are perpendicular).

   (a) From the center of circumscribed sphere around the pyramid is drawn a plane, parallel to the wall ABC, which intersects the edges OA, OB and OC respectively in the points A_1, B_1, C_1. Find the ratio between the volumes of the pyramids $OABC$ and $OA_1B_1C_1$.

   (b) Prove that if the walls OBC, OAC and OAB have the angles with the base ABC respectively α, β and γ then

$$\frac{h-r}{r} = \cos\alpha + \cos\beta + \cos\gamma$$

   where h is the distance between O and ABC plane and r is the radii of the inscribed in the pyramid $OABC$ sphere.

(8 Points)

6. Prove the equality

$$1 + \frac{\cos x}{\cos^1 x} + \frac{\cos 2x}{\cos^2 x} + \cdots + \frac{\cos nx}{\cos^n x} = \frac{\sin(n+1)x}{\sin x \cos^n x}$$

if $\cos x \neq 0$ and $\sin x \neq 0$. (5 Points)

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