Bulgarian Mathematical Olympiad 2008
Regional Round, April 19-20

Grade 9

First Day

1. Find all pairs of integer numbers (p,q) such that the roots of the equation:
$$(px-q)^2+(qx-p)^2=x$$
are also integer numbers.

2. It is given a rhombus $ABCD$ with sidelength a. On the line AC are chosen the points M and N in such a way that C lies between A and N and $MA\cdot NC=a^2$. We denote with P the intersection point of MD, BC and Q is the intersection point of ND, AB. Prove that D is the incenter of the triangle PQB.

3. Find all natural numbers n with exactly 8 natural divisors which sum is equal to 3780 (including 1 and n).

Second day

4. It is given the isosceles triangle ABC ($AC=BC$) where the angle $\angle ACB$ is equal to 30°. The point M is symmetric to the vertex A with respect to the line BC. N is symmetric to the M with respect to the vertex C. If P is the intersecting point of the lines AC and BN and Q is the intersecting point of the lines AN and PM find the ratio $AQ:QN$.

5. Solve the equation:
$$x^2-13[x]+11=0$$

6. The cities in a country are connected with paths. It is known that two cities are connected with no more than one path(s) and each city is connected with not less than three paths. A traveler left some city must pass through at least six other cities (it is not allow passing a path more than once) before he goes back to its starting position. Prove that the country have at least 24 cities.

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Grade 10

First Day

1. Find all pairs real values of x such that the following inequalities are satisfied:

$$1 \leq \sqrt{x+2} - \frac{1}{\sqrt{x+2}} \leq 4$$

2. The points A, B and C are situated on the circumference k in such a way that the tangents to k at the points A and B intersects at the point P and C lies on the bigger arc AB. Let the line through C which is perpendicular to PC intersects the line AB at the point Q. Prove that:

   (a) if the lines PC and QC intersects k for second time at the points M and N then the angles $\angle CQP$ and $\angle CMN$ are equal.

   (b) If S is the middle point of PQ then SC is tangent to k.

3. Prove that there exist a function $f : \mathbb{N} \to \mathbb{N}$, such that $f(f(n)) = 3n$ for all natural numbers n.

Second day

4. It is given the equation:

$$8^x - 2m2^x + 1 + 4m = 8$$

   where m is a real parameter.

   (a) Solve the equation for $m = 6$.

   (b) Find all real values of m for which the equation have exactly one positive solution.

5. If a, b, c are positive real numbers prove the inequality:

$$\frac{a^5}{bc} + \frac{b^5}{ca} + \frac{c^5}{ab} + \frac{3}{2a^2b^2c^2} \geq 2\left(a^3 + b^3 + c^3\right) + \frac{9}{2} - 6abc$$

   When does equality holds?

6. An isosceles trapezium with bases 1 and 5 and equal sides with length $\sqrt{7}$ is covered with 10 circles with radii r. Prove that $r \geq 1/2$. (Here *covered* means each trapezium's point is in at least one of the circles).

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Grade 11

First Day

1. It is given an arithmetic progression $a_1, a_2, \ldots, a_n, \ldots$ for which $a_1 \cdot a_2 < 0$ and
$$(a_1 + a_2 + a_3)\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}\right) = -\frac{3}{8}$$
Find the smallest natural number $n > 2$ for which $\frac{a_n}{a_2}$ is a square of natural number.

2. In the equilateral triangle ABC is chosen a point O. Its symmetric point with respect to the sides BC, CA and AB are denoted respectively with A_1, B_1 and C_1. Prove that the lines AA_1, BB_1 and CC_1 intersects at a common point.

3. Let a and b are natural numbers. Prove that the sequence: $\{a_n\}_{n=1}^{\infty}$, defined with the equalities $a_1 = a$ and $a_{n+1} = \phi(an + b)$, $n > 1$ is bounded. For every natural number k with $\phi(k)$ is denoted the number of natural numbers smaller than k and coprime with k.

Second day

4. In a circle with radii $R = 65$ is inscribed a quadrilateral $ABCD$ for which $AB = 50$, $BC = 104$ and $CD = 120$. Find the length of the side AD.

5. (a) It is given the sequence $a_n = \sqrt[n]{n}$, $n = 1, 2, \ldots$. Prove that
$$\lim_{n\to\infty} a_n = 1$$
(b) Lef $f(x)$ is a polynomial with positive integer coefficients. Prove that the sequence $b_n = \sqrt[n]{f(n)}$, $n = 1, 2, \ldots$, converges and find its limit.

6. Let k is a natural number. We denote with $f(k)$ the biggest natural number for which there exists a set M from natural numbers with $f(k)$ elements, such that:
(i) Each element from M is a divisor of k.
(ii) There are no element from M that divides some other element from M.
Prove that if m and n are coprime numbers then:
$$f(n \cdot 2^n) \cdot f(m \cdot 2^m) = f(mn \cdot 2^{m+n})$$

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Grade 12

First Day

1. In a triangle pyramid $ABCD$ adjacent edges DB and DC are equal and $\angle DAB = \angle DAC$. Find the volume of the pyramid if $AB = 15$, $BC = 14$, $CA = 13$ and $DA = 18$.

2. Find the values of real parameter a for which the graphs of the functions $f(x) = x^2 + a$ and $g(x) = x^3$ have exactly one common tangent.

3. Is there exists a natural number n for which the number $\left(\frac{2008}{n}\right)^3 + \frac{2008}{n}$ is a square of a natural number.

Second day

4. Find all natural numbers a such that
$$\left[\sqrt{n}+\sqrt{n+1}\right] = \left[\sqrt{4n+a}\right]$$
for any natural number n (with $[x]$ is denoted the integer part of the number x).

5. The incircle of ABC is tangent to the sides BC, CA and AB respectively at the points: A_1, B_1 and C_1. It is also known that the line A1B1 pass through the middle point of the segment CC_1. Find the angles of the triangle if their sines form an arithmetic progression.

6. Let $\mathbb{R}$ be the set of all real numbers. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(x+y^2) \geq (y+1)f(x)$$
for all x, y that belongs to $\mathbb{R}$.

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