Brazilian IMO & IbMO Team Selection Tests 1999

First Test – April 10, 1999

Time: 4.5 hours

1. Find all positive integers n with the following property: There exist a positive integer k and mutually distinct integers $x_1, x_2, \dots, x_n$ such that the set $\{x_i + x_j \mid 1 \le i < j \le n\}$ is a set of distinct powers of k.

2. Let a, b, c, d be real numbers such that

$$a = \sqrt{4 - \sqrt{5-a}}, \qquad b = \sqrt{4 + \sqrt{5-b}},$$
$$c = \sqrt{4 - \sqrt{5+c}}, \qquad d = \sqrt{4 + \sqrt{5+d}}.$$

   Calculate $abcd$

3. Let BD and CE be the bisectors of the interior angles $\angle B$ and $\angle C$, respectively ($D \in AC$, $E \in AB$). Consider the circumcircle of ABC with center O and the excircle corresponding to the side BC with center I_a. These two circles intersect at points P and Q.
   (a) Prove that PQ is parallel to DE.
   (b) Prove that I_aO is perpendicular to DE.

4. Let $\mathbb{Q}^+$ and $\mathbb{Z}$ denote the set of positive rationals and the set of integers, respectively. Find all functions $f : \mathbb{Q}^+ \to \mathbb{Z}$ satisfying the following conditions:
   (i) $f(1999) = 1$;
   (ii) $f(ab) = f(a) + f(b)$ for all $a, b \in \mathbb{Q}^+$;
   (iii) $f(a+b) \ge \min\{f(a), f(b)\}$ for all $a, b \in \mathbb{Q}^+$.

5. (a) If m, n are positive integers such that $2^n - 1$ divides $m^2 + 9$, prove that n is a power of 2;
   (b) If n is a power of 2, prove that there exists a positive integer m such that $2^n - 1$ divides $m^2 + 9$.

Second Test – May 15, 1999

1. For a positive integer n, let $\omega(n)$ denote the number of distinct prime divisors of n. Determine the least positive integer k such that

$$2^{\omega(n)} \le k \sqrt[4]{n}$$

   for all positive integers n.

2. In a triangle ABC, the bisector of the angle at A of a triangle ABC intersects the segment BC and the circumcircle of ABC at points A_1 and A_2, respectively. Points B_1, B_2, C_1, C_2 are analogously defined. Prove that

$$\frac{A_1A_2}{BA_2+CA_2}+\frac{B_1B_2}{CB_2+AB_2}+\frac{C_1C_2}{AC_2+BC_2}\geq\frac{3}{4}.$$

3. A sequence a_n is defined by

$$a_0=0, \qquad a_1=3;$$
$$a_n=8a_{n-1}+9a_{n-2}+16 \text{ for } n\geq 2.$$

Find the least positive integer h such that $a_{n+h}-a_n$ is divisible by 1999 for all $n\geq 0$.

4. Assume that it is possible to color more than half of the surfaces of a given polyhedron so that no two colored surfaces have a common edge.

   (a) Describe one polyhedron with the above property.

   (b) Prove that one cannot inscribe a sphere touching all the surfaces of a polyhedron with the above property.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com