# Brazilian IMO & IbMO Team Selection Tests 2004

*First Test – March 13, 2004*

1. Let $x,y,z$ be positive numbers such that $x^2+y^2+z^2=1$. Prove that
$$\frac{x}{1-x^2}+\frac{y}{1-y^2}+\frac{z}{1-z^2}\geq\frac{3\sqrt{3}}{2}.$$

2. Show that there exist infinitely many pairs of positive integers $(m,n)$ such that $\binom{m}{n-1}=\binom{m-1}{n}$.

3. Prove that there exists a family $\mathscr{F}=\{A_1,A_2,\dots,A_r\}$ of $m$-element subsets of a given set $\{b_1,b_2,\dots,b_n\}$ of $n$ elements such that

   (i) $|A_i\cap A_j|\leq m-2$ for all $A_i,A_j\in\mathscr{F}$ with $i\neq j$, and

   (ii) $r\geq\left[\frac{1}{n}\binom{n}{m}\right]$.

4. Let $I$ be the incenter of a triangle $ABC$ with $\angle BAC=60^\circ$. A line through $I$ parallel to $AC$ intersects $AB$ at $F$. Let $P$ be the point on the side $BC$ such that $3BP=BC$. Prove that $\angle BFP=\frac{1}{2}\angle ABC$.

*Second Test – April 17, 2004*

1. Find the smallest positive integer $n$ that satisfies the following condition: For every finite set of points on the plane, if for any $n$ points from this set there exist two lines containing all the $n$ points, then there exist two lines containing all points from the set.

2. Let $(x+1)^p(x-3)^q=x^n+a_1x^{n-1}+\cdots+a_{n-1}x+a_n$, where $p,q$ are positive integers.

   (a) Prove that if $a_1=a_2$, then $3n$ is a perfect square.

   (b) Prove that there exist infinitely many pairs $(p,q)$ for which $a_1=a_2$.

3. Determine the locus of points $M$ in the plane of a given rhombus $ABCD$ such that $MA\cdot MC+MB\cdot MD=AB^2$.

4. The sequence $(L_n)$ is givn by $L_0=2$, $L_1=1$ and $L_{n+1}=L_n+L_{n-1}$ for $n\geq 1$. Prove that if a prime number $p$ divides $L_{2k}-2$ for $k\in\mathbb{N}$, then $p$ also divides $L_{2k+1}-1$.

*Third Test – May 15, 2004*

1. Let $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ be distinct circles such that $\Gamma_1, \Gamma_3$ are externally tangent at $P$, and $\Gamma_2, \Gamma_4$ are externally tangent at the same point $P$. Suppose that $\Gamma_1$ and $\Gamma_2$; $\Gamma_2$ and $\Gamma_3$; $\Gamma_3$ and $\Gamma_4$; $\Gamma_4$ and $\Gamma_1$ meet at $A, B, C, D$, respectively, and that all these points are different from $P$. Prove that

$$\frac{AB \cdot BC}{AD \cdot DC} = \frac{PB^2}{PD^2}.$$

2. An integer $n \geq 2$ is called *amicable* if there exist subsets $A_1, A_2, \dots, A_n$ of the set $\{1, 2, \dots, n\}$ such that

   (i) $i \notin A_i$ for any $i = 1, 2, \dots, n$,

   (ii) $i \in A_j$ if and only if $j \notin A_i$, for any $i \neq j$,

   (iii) $A_i \cap A_j \neq \emptyset$ for any $i, j \in \{1, \dots, n\}$.

   (a) Prove that 7 is amicable.

   (b) Prove that $n$ is amicable if and only if $n \geq 7$.

3. Set $\mathbb{Q}_1 = \{x \in \mathbb{Q} \mid x \geq 1\}$. Suppose that a function $f : \mathbb{Q}_1 \to \mathbb{R}$ satisfies the inequality $|f(x+y) - f(x) - f(y)| < \varepsilon$ for all $x, y \in \mathbb{Q}_1$, where $\varepsilon > 0$ is given. Prove that there exists a real number $q$ such that

$$\left| \frac{f(x)}{x} - q \right| < 2\varepsilon \quad \text{for all } x \in \mathbb{Q}_1.$$

4. Let $b$ be an integer greater than 5. For each positive integer $n$, consider the number

$$x_n = \underbrace{11\dots1}_{n-1}\underbrace{22\dots2}_{n}5,$$

   written in base $b$. Prove that the following condition holds if and only if $b = 10$: There exists a positive integer $M$ such that for every integer $n$ greater than $M$, the number $x_n$ is a perfect square.



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