Brazilian IMO & IbMO Team Selection Tests 2001

First Test – March 24, 2001

Time: 4.5 hours

1. Find all functions \( f : \mathbb{R} \to \mathbb{R} \) satisfying
\[
f(x+y) + f(y+z) + f(z+x) \geq f(x+2y+3z)
\]
for all real \( x, y, z \).

2. Let \( f(n) \) be the least positive integer \( k \) such that \( n \) divides \( 1 + 2 + \cdots + k \). Prove that \( f(n) = 2n - 1 \) if and only if \( n \) is a power of 2.

3. For which positive integers \( n \) is there a permutation \( (x_1, x_2, \ldots, x_n) \) of \( 1, 2, \ldots, n \) such that all the differences \( |x_k - k|, k = 1, 2, \ldots, n \), are distinct?

4. Let \( ABC \) be a triangle with the circumcenter at \( O \). Let \( P, Q \) be points on the segments \( AB \) and \( AC \) respectively so that
\[
BP : PQ : QC = AC : CB : BA.
\]
Prove that the points \( A, P, Q \) and \( O \) are concyclic.

Second Test – May 19, 2001

1. Polynomials \( P(x) \) and \( Q(x) \) with real coefficients, both of which having at least one real root, satisfy the equality
\[
P(1 + x + Q(x)^2) = Q(1 + x + P(x)^2)
\]
for all real \( x \). Prove that the polynomials \( P \) and \( Q \) are equal.

2. A set \( S \) consists of \( k \) sequences of 0, 1, 2 of length \( n \). For any two sequences \( (a_i), (b_i) \in S \) we can construct a new sequence \( (c_i) \) such that \( c_i = \left\lfloor \frac{a_i + b_i + 1}{2} \right\rfloor \) and include it in \( S \). Assume that after performing finitely many such operations we obtain all the \( 3^n \) sequences of 0, 1, 2 of length \( n \). Find the least possible value of \( k \).

3. Let \( ABC \) be a triangle and \( D, E \) be the points of intersection of the internal and external bisectors of the angle at \( A \) with \( BC \). Let \( F \neq A \) be the intersection point of line \( AC \) with the circle with diameter \( DE \). Let \( G \neq A \) be the point at which the tangent at \( A \) on the circumcircle of \( ABF \) meets the circle with diameter \( DE \). Prove that \( AF = AG \).

4. Prove that for all integers \( n \geq 3 \) there exists a set \( A_n = \{a_1, a_2, \ldots, a_n\} \) of \( n \) distinct natural numbers such that, for each \( i = 1, 2, \ldots, n \),
\[
\prod_{1 \leq j \leq n \atop j \neq i} a_k \equiv 1 \mod a_i.
\]